Invariant domain preserving discretization-independent schemes and convex limiting for hyperbolic systems

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ARL

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Outline

Hyperbolic systems

The PDEs

- Hyperbolic system

$$\partial_t \mathbf{u} + \nabla\cdot\mathbb{f}(\mathbf{u}) = \mathbf{S}(\mathbf{u}), \qquad (\mathbf{x}, t) \in D\times\mathbb{R}_+.$$
$$u(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \qquad \mathbf{x} \in D.$$

- D open polyhedral domain in $\mathbb{R}^d$.
- $\mathbb{f} \in \mathcal{C}^1(\mathbb{R}^m; \mathbb{R}^{m\times d})$, the flux. $\mathbf{S} : \mathbb{R}^m \to \mathbb{R}^m$ source term
- $\mathbf{u}_0$, admissible initial data.
- Periodic BCs or $\mathbf{u}_0$ has compact support (to simplify BCs)

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Formulation of the problem

Assumptions for the hyperbolic part

- $\exists$ admissible set $\mathcal{A} \subset \mathbb{R}^m$ s.t. for all $(\mathbf{u}_l, \mathbf{u}_r) \in \mathcal{A}$ the 1D Riemann problem

$$\partial_t \mathbf{v} + \partial_x(\mathbb{f}(\mathbf{v})\mathbf{n}) = 0, \quad \mathbf{v}(x,0) = \begin{cases} \mathbf{u}_l & \text{if } x < 0 \\ \mathbf{u}_r & \text{if } x > 0. \end{cases}$$

  has a unique "entropy" solution $\mathbf{u}(\mathbf{u}_l, \mathbf{u}_r)(x,t)$ for all $\mathbf{n} \in \mathbb{R}^d$, $\|\mathbf{n}\|_{\ell^2} = 1$.
- There exists an invariant set $\mathcal{B} \subset \mathcal{A} \subset \mathbb{R}^m$, i.e.,

$$((\mathbf{u}_l, \mathbf{u}_r) \in \mathcal{B}) \Rightarrow (\mathbf{v}(\mathbf{u}_l, \mathbf{u}_r)(x,t) \in \mathcal{B}, \forall t \geq 0, \forall x \in \mathbb{R})$$

- $\mathcal{B}$ is convex.

Assumptions for source term

- $\mathcal{B}$ is invariant w.r.t. $\mathbf{S}$

$$\exists \Delta t_0, \qquad (\mathbf{u} \in \mathcal{B}) \Rightarrow (\mathbf{u} + \Delta t \mathbf{S}(\mathbf{u}) \in \mathcal{B}, \quad \forall \Delta t \leq \Delta t_0)$$

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Strategy for high-order method

(i) Construct low-order invariant domain preserving method.

(ii) Construct a high-order scheme that may not be invariant domain preserving.

(iii) Apply convex limiting with correct bounds inferred from low-order solution to get a high-order method that is invariant domain preserving.

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Invariant domain preserving discretization

- Current time t^n; time step Δt
- Discrete scalar space $X_h = \text{span}\{\varphi_i\}_{i\in\mathcal{V}}$
- Vector-valued space $\mathbf{X}_h = (X_h)^m$
- $\mathbf{u}_h(\cdot, t^n)$ approximated by $\sum_{i\in\mathcal{V}} \mathbf{U}_i^n \varphi_i$, $\mathbf{U}_i^n \in \mathbb{R}^m$
- Assume high-order approximation of $\mathbf{u}(\cdot, t^{n+1})$ as follows:

$$\frac{m_i}{\Delta t}(\mathbf{U}_i^{\mathrm{G},n+1} - \mathbf{U}_i^n) + \sum_{j\in\mathcal{I}(i)} \mathbb{f}(\mathbf{U}_j^n)\mathbf{c}_{ij} = m_i \mathbf{S}_i^n,$$

- Here $\mathcal{I}(i)$ is the local stencil. List of dofs interacting with i.
- Think of above approximation as Galerkin approximation, or centered approximation, or inviscid approximation. (No stabilization, no artificial viscosity.)

Discretization-independent invariant domain approximation

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Assumptions

- Lumped mass is positive

$$m_i > 0, \qquad \forall i \in \mathcal{V}$$

- Conservation

$$\mathbf{c}_{ij} = -\mathbf{c}_{ij}, \qquad \text{and} \qquad \sum_{j\in\mathcal{I}(i)} \mathbf{c}_{ij} = 0.$$

Lemma

- *Conservation*

$$\sum_{i\in\mathcal{V}} m_i \mathbf{U}_i^{\mathrm{G},n+1} = \sum_{i\in\mathcal{V}} m_i \mathbf{U}_i^n.$$

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Key idea

- Consider undirected graph $(\mathcal{V}, \mathcal{E})$, $((i,j) \in \mathcal{E}) \Leftrightarrow (i \in \mathcal{I}(j),\ \text{and}\ j \in \mathcal{I}(i))$.
- Introduce artificial viscosity using graph Laplacian on $(\mathcal{V}, \mathcal{E})$.

Guaranteed Maximum Speed Graph Viscosity (GMS-GV)

$$\frac{m_i}{\Delta t}(\mathbf{U}_i^{\mathrm{L},n+1} - \mathbf{U}_i^n) + \sum_{j\in\mathcal{I}(i)} \mathbb{f}(\mathbf{U}_j^n)\mathbf{c}_{ij} - \sum_{j\in\mathcal{I}(i)\setminus\{i\}} d_{ij}^{\mathrm{L},n}(\mathbf{U}_j^n - \mathbf{U}_i^n) = m_i\mathbf{S}(\mathbf{U}_i^n),$$

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GMS-GV scheme

Assumptions

- Graph viscosity is positive

$$d_{ij}^{\mathsf{L},n} \geq 0, \qquad i \neq j$$

- Conservation. Assume

$$d_{ij}^{\mathsf{L},n} = d_{ij}^{\mathsf{L},n}.$$

Lemma

- *Symmetry* $\boxed{\mathbf{c}_{ij} = -\mathbf{c}_{ij} \text{ and } d_{ij} = d_{ji}}$ *implies conservation*

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Key ideas (Lax, Perthame, Tadmor, Shu, etc.)

- Let $i \in \mathcal{V}$, $j \in \mathcal{I}(i)$.
- Define unit vector $\mathbf{n}_{ij} := \frac{\mathbf{c}_{ij}}{\|\mathbf{c}_{ij}\|_{\ell^2}}$
- Introduce $\lambda_{\max}(\mathbf{n}_{ij}, \mathbf{U}_i^n, \mathbf{U}_j^n)$, upper bound on maximum wave speed in 1D Riemann problem:

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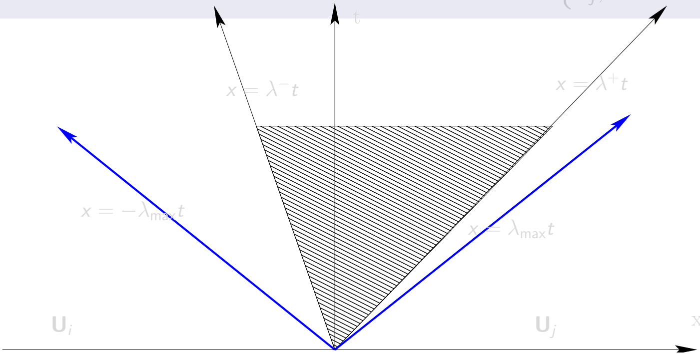

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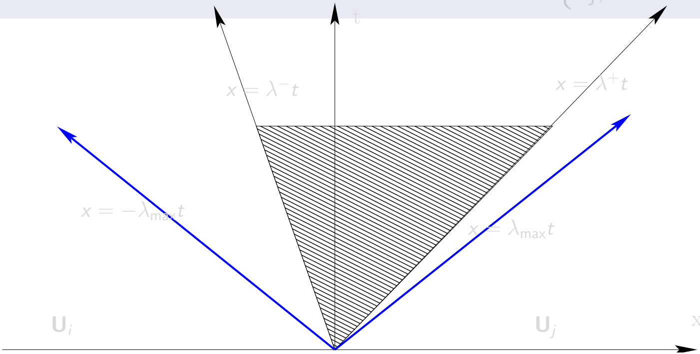

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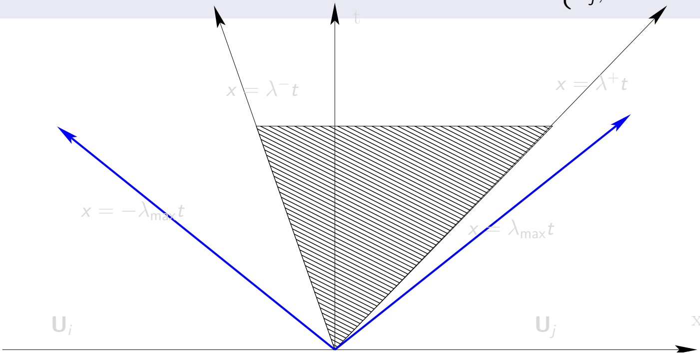

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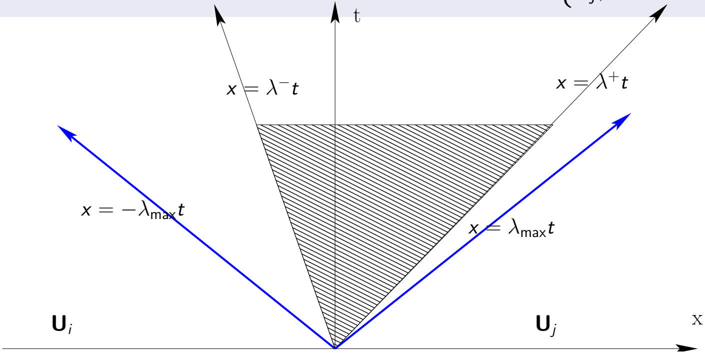
- Introduce $\lambda_{\max}(\mathbf{n}_{ij}, \mathbf{U}_i^n, \mathbf{U}_j^n)$, upper bound on maximum wave speed in 1D Riemann problem:

$$\partial_t \mathbf{u} + \partial_x(\mathbb{f}(\mathbf{u})\mathbf{n}) = 0, \quad (x,t) \in \mathbb{R}\times\mathbb{R}_+, \qquad \mathbf{u}(x,0) = \begin{cases} \mathbf{U}_i^n, & \text{if } x < 0 \\ \mathbf{U}_j, & \text{if } x > 0, \end{cases}$$



Key ideas

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$$d_{ij}^{\mathrm{L},n} := \max(\lambda_{\max}(\mathbf{n}_{ij}, \mathbf{U}_i^n, \mathbf{U}_j^n)\|\mathbf{c}_{ij}\|_{\ell^2}, \lambda_{\max}(\mathbf{n}_{ji}, \mathbf{U}_j^n, \mathbf{U}_i^n)\|\mathbf{c}_{ji}\|_{\ell^2}).$$

- Introduce auxiliary states ("bar states")

$$\overline{\mathbf{U}}_{ij}^n := \frac{1}{2}(\mathbf{U}_i^n + \mathbf{U}_j^n) - (\mathbb{f}(\mathbf{U}_j^n) - \mathbb{f}(\mathbf{U}_i^n))\frac{\mathbf{c}_{ij}}{2d_{ij}^{\mathrm{L},n}}.$$

Lemma (Invariance of the auxiliary states, JLG+BP (2016-2018))

Let $\mathcal{B} \subset \mathcal{A}$ be any convex invariant set such that $\mathbf{U}_i^n, \mathbf{U}_j^n \in \mathcal{B}$. Then, the state $\overline{\mathbf{U}}_{ij}^n$ belongs to $\mathcal{B}$.

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GMS-GV scheme

Key observation

- Convex combination

$$\mathbf{U}_i^{\mathrm{L},n+1} = \frac{1}{2}\left(\left(1 + 4\Delta t \frac{d_{ii}^{\mathrm{L},n}}{m_i}\right)\mathbf{U}_i^n + \sum_{j\in\mathcal{I}(i)\setminus\{i\}} \frac{4\Delta t d_{ij}^{\mathrm{L},n}}{m_i}\overline{\mathbf{U}}_{ij}^n\right) + \frac{1}{2}(\mathbf{U}_i^n + 2\Delta t \mathbf{S}(\mathbf{U}_i^n)).$$

Theorem (GMS-GV, Local invariance, JLG+BP (2016-2018))

Let $n \geq 0$ and let $i \in \mathcal{V}$. Assume that Δt is small enough so that $1 - 4\Delta t \frac{\sum_{j\in\mathcal{I}(i)\setminus\{i\}} d_{ij}^{\mathrm{L},n}}{m_i} \geq 0$ and $2\Delta t \leq \tau_0$. Let $\mathcal{B} \subset \mathcal{A}$ be a convex invariant set such that $\mathbf{U}_j^n \in \mathcal{B}$ for all $j \in \mathcal{I}(i)$, then $\mathbf{U}_i^{\mathrm{L},n+1} \in \mathcal{B}$.

- This is the generalization of the maximum principle for any discretization (any mesh), in any space dimension, for any hyperbolic system.
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Literature: **Lax (1954)**, **Chueh, Conley, Smoller (1973)**, **Hoff (1979, 1985)**, **Perthame-Shu (1996)**, **Frid (2001)**, **Zhang-Shu (2010), (2011)**, ...

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Example (Continuous FE)

- $\{\mathcal{T}_h\}_{h>0}$ shape regular conforming mesh sequence
- Shape functions $\{\varphi_1, \ldots, \varphi_I\}$ + partition of unity ($\sum_{j\in\{1:I\}} \varphi_j = 1$)
- Ex: $\mathbb{P}_1$, $\mathbb{Q}_1$, Bernstein polynomials (any degree)
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Example 1: Continuous finite elements

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Example

Finite volumes

$$\boxed{m_i := |K_i|}, \quad \boxed{\mathbf{c}_{ij} := \frac{|\Gamma_{ij}|}{2}\mathbf{n}_{ij}}, \; \forall j \in \mathcal{I}(i)\backslash\{i\}, \quad \mathbf{c}_{ii} := \mathbf{0}.$$

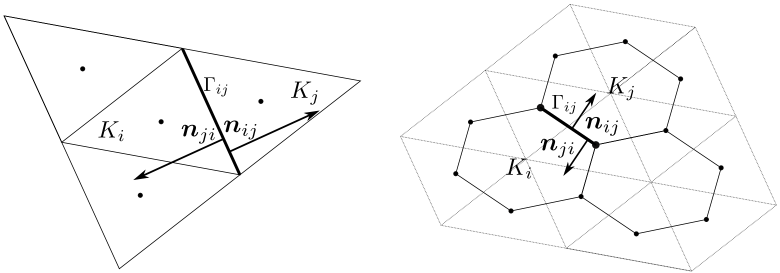

Figure: Finite volume patch arising from a cell-centered discretization (left) and a vertex-centered discretization (right).

Outline

High-order viscosity

High-order viscosity (II)

- Key idea: measure smoothness of an entropy using the chain rule.

$$\left(\nabla \mathbf{F}(\mathbf{u}) = (\nabla \eta(\mathbf{u}))^\mathsf{T} \nabla \mathbb{f}(\mathbf{u})\right) \Rightarrow \boxed{\nabla\cdot(\mathbf{F}(\mathbf{u})) = (\nabla \eta(\mathbf{u}))^\mathsf{T} \nabla\cdot(\mathbb{f}(\mathbf{u}))}$$

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High-order viscosity (II): Commutator-based entropy viscosities

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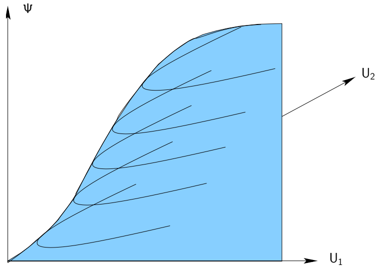
Outline

Convex limiting

Quasiconcavity



Definition (Quasiconcavity)

Given a convex set $\mathcal{B} \subset \mathbb{R}^m$, we say that a function $\Psi : \mathcal{B} \to \mathbb{R}$ is quasiconcave if the set $\{\mathbf{U} \in \mathcal{B};\ \Psi(\mathbf{U}) \geq \chi\}$ is convex for any $\chi \in \mathbb{R}$.

Quasiconcavity

Strategy

- Let $\Psi : \mathcal{B} \to \mathbb{R}$ be a quasiconcave functional.
- If low-order update satisfies $\Psi(\mathbf{U}_i^{\mathrm{L},n+1}) \geq 0$, then we want to "limit" the high-order update $\mathbf{U}_i^{\mathrm{H},n+1} \to \mathbf{U}_i^{n+1}$ so that $\boxed{\Psi(\mathbf{U}_i^{n+1}) \geq 0}$.

Fundamental questions

- What should be limited?
- How can that be done so that the result is guaranteed.

Quasiconcavity

Strategy

- Let $\Psi : \mathcal{B} \to \mathbb{R}$ be a quasiconcave functional.
- If low-order update satisfies $\Psi(\mathbf{U}_i^{\mathrm{L},n+1}) \geq 0$, then we want to "limit" the high-order update $\mathbf{U}_i^{\mathrm{H},n+1} \to \mathbf{U}_i^{n+1}$ so that $\boxed{\Psi(\mathbf{U}_i^{n+1}) \geq 0}$.

Fundamental questions

- What should be limited?
- How can that be done so that the result is guaranteed.

Example

- The negative of every entropy is quasiconcave (concave, actually).
- The negative of every specific entropy is quasiconcave (not concave).
- The negative of kinetic energy is quasiconcave (not concave).
- Internal energy is quasiconcave (concave, actually).
- Specific internal energy is quasiconcave (not concave).
- Every scalar component of **U** is quasiconcave (concave and convex, actually). (Think density, water height, etc.)

Example (with matrices, $\mathbb{R}^m = \mathbb{R}^{k^2}$, $m = k^2$.)

- Log-determinant of SPD matrices is concave.
- Negative of largest eigenvalue of SPD matrices is concave.

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Lemma (Quasiconcavity)

Let $\mathcal{B} \subset \mathbb{R}^m$ be convex set.

$\Psi : \mathcal{B} \to \mathbb{R}$ *is quasiconcave*

$\Updownarrow$

for every finite set $\mathcal{S} \subset \mathbb{N}$, every set of convex coefficients $\{\lambda_j\}_{j \in \mathcal{S}}$ (i.e., $\sum_{j \in \mathcal{S}} \lambda_j = 1$ and $\lambda_j \geq 0$ for all $j \in \mathcal{S}$), and every collection of vectors $\{\mathbf{U}_j\}_{j \in \mathcal{S}}$ in $\mathcal{B}$, the following holds true:

$$\Psi\Big(\sum_{j \in \mathcal{S}} \lambda_j \mathbf{U}_j\Big) \geq \min_{j \in \mathcal{S}} \Psi(\mathbf{U}_j).$$

Lemma (Fundamental bounds on the GMS-GV scheme, JLG+BP+IT (2018))

- *Let $\mathcal{B} \subset \mathcal{A} \subset \mathbb{R}^m$ be any convex set.*
- *Let $\Psi : \mathcal{B} \to \mathbb{R}$ be any quasiconcave functional.*
- *Let $n \geq 0$, $i \in \mathcal{V}$. Assume that $1 - 4\Delta t \frac{\sum_{j\in\mathcal{I}(i)} d_{ij}^{\mathrm{L},n}}{m_i} \geq 0$ and $2\Delta t \leq \tau_0$.*
- *Let $\{\overline{\mathbf{U}}_{ij}^n\}_{j\in\mathcal{I}(i)}$ be the auxiliary states. Define the following quantity:*

$$\boxed{\Psi_i^{\min} := \min\Big(\min_{j\in\mathcal{I}(i)} \Psi(\overline{\mathbf{U}}_{ij}^n), \Psi(\mathbf{U}_i^n + 2\Delta t\mathbf{S}(\mathbf{U}_i^n))\Big)}$$

- *Assume that $\mathbf{U}_j^n \in \mathcal{B}$ for all $j \in \mathcal{I}(i)$.*

Then, the first-order update $\mathbf{U}_i^{\mathrm{L},n+1}$ computed with the GMS-GV scheme satisfies the following inequality:

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Key principles

- Let $\mathbf{U}^n$ be current solution, $n \geq 0$.
- Compute invariant domain preserving low-order solution $\mathbf{U}^{\mathrm{L},n+1}$.
- Consider family of quasi-concave functionals $\Psi^l : \mathcal{B}^l \to \mathbb{R}$, $l \in \mathcal{L}$.
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- Observe that $\boxed{\Psi_i^l(\mathbf{U}_i^{\mathrm{L},n+1}) \geq 0}$
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Limiter set in not empty because $\ell_{ij} = 0$ is an admissible limiter.

The above program is meaningful!

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- Difference, compute limiter $\ell_{ij} = \ell_{ji} \in [0,1]$ as large as possible s.t.:

$$\boxed{\Psi_i^l(\mathbf{U}_i^{\mathrm{L},n+1} + \sum_{j\in\mathcal{I}(i)\backslash\{i\}} \ell_{ij}\mathbf{A}_{ij}^n) \geq 0}, \qquad \mathbf{A}_{ij}^n = \frac{\Delta t}{m_i}(\mathbf{F}_{ij}^{\mathrm{H},n} - \mathbf{F}_{ji}^{\mathrm{L},n}).$$

Lemma (Limiter set not empty)

Limiter set in not empty because $\ell_{ij} = 0$ is an admissible limiter.

The above program is meaningful!

Abstract framework

Strategy

Divide and conquer: Take care of each pair (i, j) separately.

Lemma

- *Let* $\{\lambda_j\}_{j\in\mathcal{I}(i)\setminus\{i\}}$ *be any set of strictly positive convex coefficients.*
- *Let* $\mathbf{P}_{ij}^n := \frac{1}{\lambda_j}\mathbf{A}_{ij}^n$.
- *Let* $\Psi_i^l : \mathcal{B} \to \mathbb{R}$ *be a quasiconcave function.*
- *Assume that the limiting parameters* $\ell_{ij} \in [0,1]$ *are such that* $\Psi_i^l(\mathbf{U}_i^{\mathrm{L},n+1} + \ell_{ij}\mathbf{P}_{ij}^n) \geq 0$, *for all* $j \in \mathcal{I}(i)\setminus\{i\}$.

Then the following inequality holds true:

$$\Psi_i^l(\mathbf{U}_i^{n+1}) := \Psi_i^l\Bigg(\sum_{j\in\mathcal{I}(i)\setminus\{i\}} \lambda_j(\mathbf{U}_i^{\mathrm{L},n+1} + \ell_{ij}\mathbf{P}_{ij}^n)\Bigg) \geq 0$$

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Example

Convex coefficients $\{\lambda_j\}_{j\in\mathcal{I}(i)\setminus\{i\}}$

- Example 1: $\lambda_j = \frac{1}{\text{card}(\mathcal{I}(i)\setminus\{i\})}$.
- Example 2: $\lambda_j = \frac{m_{ij}}{\sum_{k\in\mathcal{I}(i)\setminus\{i\}} m_{ik}}$.
- Example 3: $\lambda_j = \frac{\|\mathbf{c}_{ij}\|_{\ell^2}}{\sum_{k\in\mathcal{I}(i)\setminus\{i\}} \|\mathbf{c}_{ik}\|_{\ell^2}}$.
- Example 4: symmetry-preserving, linearity-preserving, etc. (a lot of choices to explore)

Theorem (JLG+BP+IT (2018))

For every $i \in \mathcal{V}$ and $j \in \mathcal{I}(i)$, let ℓ_j^i be defined by

$$\ell_j^i = \begin{cases} 1 & \text{if } \Psi_i(\mathbf{U}_i^{\mathrm{L},n+1} + \mathbf{P}_{ij}^n) \geq 0, \\ \max\{\ell \in [0,1];\ \Psi_i(\mathbf{U}_i^{\mathrm{L},n+1} + \ell \mathbf{P}_{ij}^n) \geq 0\} & \text{otherwise.} \end{cases}$$

The following statements hold true:

1. ℓ_j^i *is uniquely defined.*
2. $\Psi_i(\mathbf{U}_i^{\mathrm{L},n+1} + \ell \mathbf{P}_{ij}^n) \geq 0$ *for every* $\ell \in [0, \ell_j^i]$;
3. *Setting* $\ell_{ij} = \min(\ell_j^i, \ell_i^j)$, *we have* $\Psi_i(\mathbf{U}_i^{\mathrm{L},n+1} + \ell_{ij} \mathbf{P}_{ij}^n) \geq 0$ *and* $\ell_{ij} = \ell_{ji}$.

Take home message

- For every i, limiter computed for every pair (i,j) in the stencil of i.
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Abstract framework

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Implementation

Claim

- If $\ell \to \Psi_i(\mathbf{U}^{\mathrm{L}} + \ell\mathbf{P})$ is linear or quadratic, $\Rightarrow$ line search $=$ solving linear or quadratic equation.
- If $\ell \to \Psi_i(\mathbf{U}^{\mathrm{L}} + \ell\mathbf{P})$ is concave for all $\ell \in [0, \ell_{\mathrm{max}}]$, $\Rightarrow$ line search can be done very efficiently with Newton-Secant algorithm.
- Alg. always return limiter ℓ^{out} that satisfies $\Psi_i(\mathbf{U}^{\mathrm{L}} + \ell^{\mathrm{out}}\mathbf{P}) > 0$ for any tolerance.

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Numerical illustrations

Numerical illustrations

- 1D smooth wave (Euler+γ-law) in $D = (0, 1)$.
- Solution: $v(x,t) = 1$, $p(x,t) = 1$ and

$$\rho(x,t) = \begin{cases} 1 + 2^6 (x_1 - x_0)^{-6} (x - t - x_0)^3 (x_1 - x + t)^3 & \text{if } x_0 \le x - t < x_1 \\ 1 & \text{otherwise} \end{cases}$$

Table: $\mathbb{P}_1$ meshes, Convergence tests with Code 1(Entropy commutator GV) and Code 3(smoothness-based GV), CFL $= 0.25$.

# nodes	Code 1	
	δ_∞	rate
100	9.02E-03	
200	1.34E-04	6.07
400	1.01E-05	3.72
800	1.12E-06	3.18
1600	1.23E-07	3.19
3200	1.33E-08	3.21
6400	1.42E-09	3.22

2D isentropic vortex (Euler+γ-law)

- $\rho_\infty = P_\infty = T_\infty = 1$, $\mathbf{u}_\infty = (u_\infty, v_\infty)^\mathsf{T}$, $u_\infty = 1$, $v_\infty = 1$.

$$\rho(\mathbf{x},t) = (T_\infty + \delta T)^{1/(\gamma-1)}, \qquad \mathbf{u}(\mathbf{x},t) = \mathbf{u}_\infty + \delta\mathbf{u}, \qquad p(\mathbf{x},t) = \rho^\gamma$$

$$\delta\mathbf{u}(\mathbf{x},t) = \frac{\beta}{2\pi} e^{\frac{1-r^2}{2}}(-\bar{x}_2, \bar{x}_1), \qquad \delta T(\mathbf{x},t) = -\frac{(\gamma-1)\beta^2}{8\gamma\pi^2} e^{1-r^2},$$

- With $\bar{\mathbf{x}} = (x_1 - x_1^0 - u_\infty t, x_2 - x_2^0 - v_\infty t)$, $r^2 = \|\bar{\mathbf{x}}\|^2_{\ell^2}$, $\gamma = \frac{7}{5} = 1.4$, $\beta = 5$, $(x_1^0, x_2^0) = (4,4)$.

Table: Isentropic vortex, $\mathbb{P}_1$ meshes, convergence tests with limiting and without limiting, $t = 2$. Code 2, CFL $= 0.1$.

	# nodes	$\delta_1(t)$	rate	$\delta_2(t)$	rate	$\delta_\infty(t)$	rate
Limiting	441	2.46E-02	–	9.61E-02	–	1.31E+00	–
	1681	8.50E-03	1.59	3.36E-02	1.57	4.45E-01	1.62
	6561	1.99E-03	2.13	7.45E-03	2.21	9.40E-02	2.28
	25921	4.16E-04	2.28	1.50E-03	2.33	2.05E-02	2.21
	103041	8.11E-05	2.37	2.92E-04	2.37	4.68E-03	2.14
No limiting	441	2.47E-02	–	9.61E-02	–	1.31E+00	–
	1681	8.48E-03	1.60	3.34E-02	1.58	4.52E-01	1.59
	6561	1.99E-03	2.13	7.45E-03	2.20	9.40E-02	2.31
	25921	4.16E-04	2.28	1.50E-03	2.33	2.05E-02	2.21
	103041	8.11E-05	2.37	2.92E-04	2.37	4.68E-03	2.14

Usual suspects (Continuous $\mathbb{P}_1$ finite elements)

# nodes	Code 3		Low-order	
	$\delta_1(t)$	rate	$\delta_1(t)$	rate
100	1.49E-01		2.61E-01	
200	9.01E-02	0.72	1.94E-01	0.43
400	4.92E-02	0.87	1.41E-01	0.46
800	2.61E-02	0.91	9.95E-02	0.50
1600	1.34E-02	0.96	6.74E-02	0.56
3200	6.83E-03	0.97	4.40E-02	0.62
6400	3.42E-03	0.99	2.78E-02	0.66
12800	1.70E-03	1.00	1.73E-02	0.68

Table: Leblanc shocktube, $\mathbb{P}_1$ meshes, Convergence tests with Code 1 and Code 3, CFL $= 0.25$.

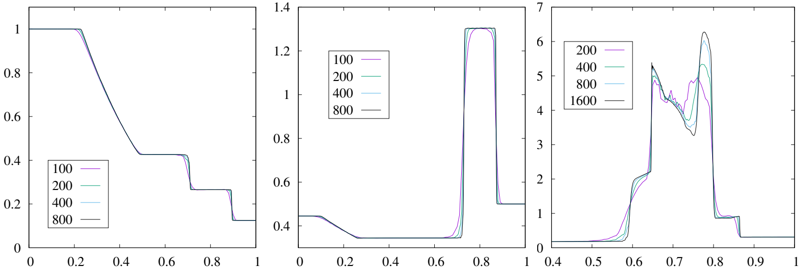

Figure: Left: Sod shocktube; Center: Lax shocktube; Right: Woodward-Collela blast wave.

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1600	1.34E-02	0.96	6.74E-02	0.56
3200	6.83E-03	0.97	4.40E-02	0.62
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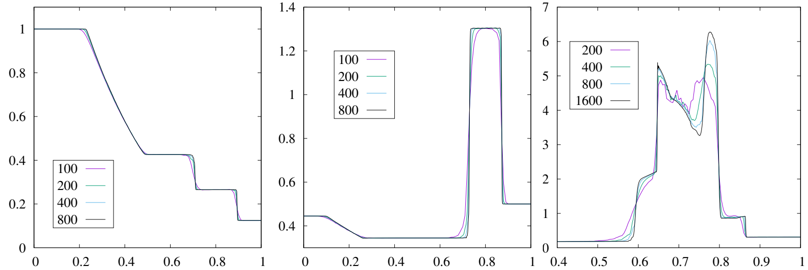


Figure: Left: Sod shocktube; Center: Lax shocktube; Right: Woodward-Collela blast wave.

Forward facing step, Mach 3 (Continuous $\mathbb{P}_1$ and $\mathbb{Q}_1$ finite elements)

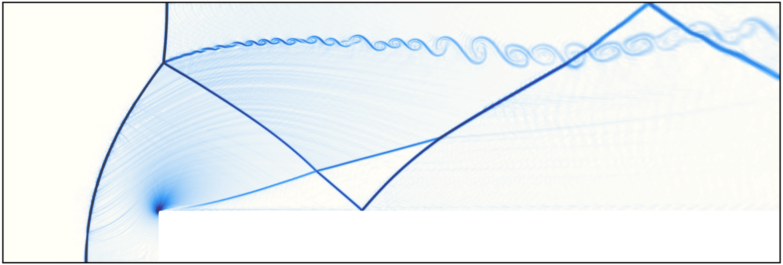


Figure: Mach 3 step, $t = 4$, density. Top: unstructured grid, 207340 $\mathbb{P}_1$ nodes. Limiting done on density (local min and max) and specific entropy (local min).

Cylinder in a channel (Continuous $\mathbb{P}_1$ finite elements)

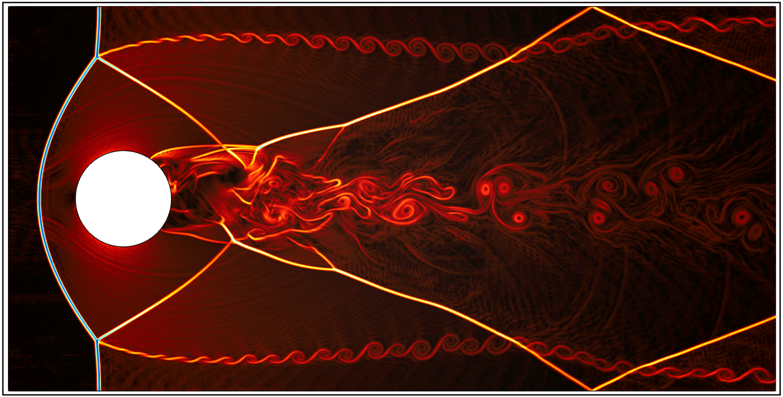
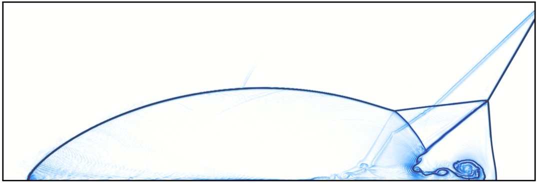


Figure: Cylinder in a channel 366764 $\mathbb{P}_1$ nodes. Limiting done on density (local min and max) and specific entropy (local min).

▸ Cylinder at Mach 3

Double Mach reflection, Mach 10 (Continuous $\mathbb{P}_1$ and $\mathbb{Q}_1$ finite elements)



Limiting done on density (local min and max) and specific entropy (local min).

Mach 10

Outline

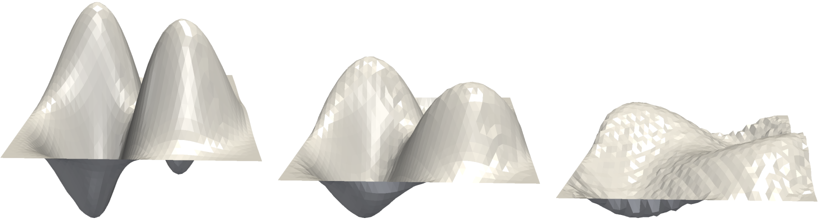
Beyond second-order

First-order solution: Dependence w.r.t polynomial degree k

- Swirling flow: $\partial_t u + \boldsymbol{\beta}\cdot\nabla u = 0$

$$\boldsymbol{\beta}(\mathbf{x},t) = \cos(\pi t)\big(-\sin(2\pi x_2)\sin^2(\pi x_1)\mathbf{e}_1 + \sin(2\pi x_1)\sin^2(\pi x_2)\mathbf{e}_2\big)$$

- Solution is periodic in time with period 1.
- Initial data: $u(x,t) = 2^8(x_1(1-x_1)x_2(1-x_2))^2\sin(2\pi x_1)\sin(2\pi x_2)$.



$\mathbb{P}_1$, 1927 nodes, $\mathbb{P}_2$ 1945 nodes, $\mathbb{P}_3$ 1888 nodes

- Origin of the problem: Size of stencil increases with k.

First-order solution: Dependence w.r.t polynomial degree k

- Swirling flow: $\partial_t u + \boldsymbol{\beta}\cdot\nabla u = 0$

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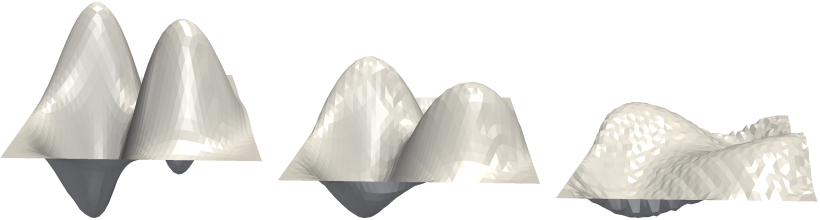

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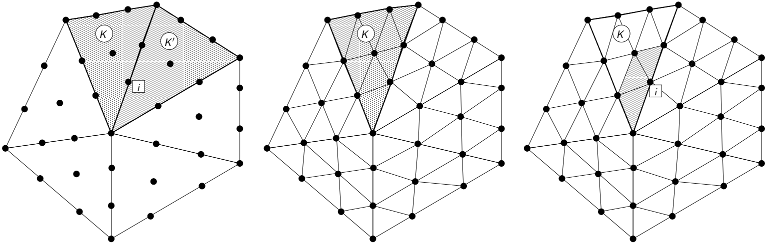
Hierarchical structure



$\mathcal{T}(i)$ $\mathcal{T}^{\mathrm{s}}_K$ $\mathcal{T}^{\mathrm{s}}_{K,i}$

Hierarchical structure

- Introduce mesh $\mathcal{T}_h$
- Introduce high-order finite element $(\widehat{K}, \widehat{P}^{\rm H}, \widehat{\Sigma}^{\rm H})$ and high-order space

$$boxed P^{\rm H}(\mathcal{T}_h) := \{v \in C^0(D;\mathbb{R});\ v_{|K}\circ T_K \in \widehat{P}^{\rm H},\ \forall K \in \mathcal{T}_h\}$$

- Not necessarily Lagrange elements (could be Bernstein).

- Introduce subdivided mesh $\mathcal{T}_h^{\rm s}$
- Introduce low-order Lagrange finite element $(\widehat{K}, \widehat{P}^{\rm L}, \widehat{\Sigma}^{\rm L})$ and low-order space

$$boxed P^{\rm L}(\mathcal{T}_h) := \{v \in C^0(D;\mathbb{R});\ v_{|K}\circ T_T \in \widehat{P}^{\rm L},\ \forall T \in \mathcal{T}_h^{\rm s}\}$$

- Introduce mesh $\mathcal{T}_h$
- Introduce high-order finite element $(\widehat{K}, \widehat{P}^{\text{H}}, \widehat{\Sigma}^{\text{H}})$ and high-order space

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Assumptions

- (If $P^{\mathrm{H}}(\mathcal{T}_h)$ is a Lagrange space) For all $K \in \mathcal{T}_h$, there exists $\{\beta^T\}_{T\in\mathcal{T}_K^{\mathrm{s}}}$ s.t. the following holds true for every $\mathbf{p} \in \mathbf{P}_K^{\mathrm{H}} := \{\widehat{\mathbf{p}}\circ\mathbf{T}_K^{-1};\, \widehat{\mathbf{p}} \in \widehat{\mathbf{P}}^{\mathrm{H}}\}$:

$$\int_K \nabla\cdot(\mathbf{p})\,\mathrm{d}x = \sum_{T\in\mathcal{T}_K^{\mathrm{s}}} \beta^T \int_T \nabla\cdot(\Pi_T^{\mathrm{Lag}}(\mathbf{p}))\,\mathrm{d}x.$$

- For all $K \in \mathcal{T}_h$ and all $T \in \mathcal{T}_K^{\mathrm{s}}$, there exists $\{\alpha_j^T\}_{j\in\mathcal{I}^{\mathrm{L}}(T)}$ s.t.

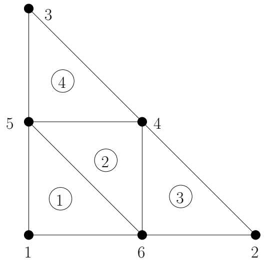
$$\forall j \in \mathcal{I}(K),\quad \sum_{T\in\mathcal{T}_{K,j}^{\mathrm{s}}} \alpha_j^T = 1,$$

$$\forall T \in \mathcal{T}_K^{\mathrm{s}},\quad \sum_{j\in\mathcal{I}^{\mathrm{L}}(T)} \alpha_j^T m_j^K = \beta^T |T|.$$

Literature

Hierarchical structure similar to high-order Residual Distribution method (**Abgrall et al. (2017)**)

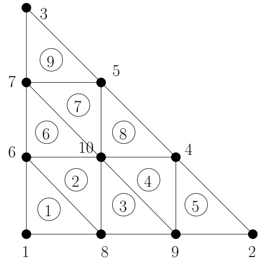
Hierarchical structure $\mathbb{P}_2$ in 2D



l	1	2	3	4
$\widehat{\beta}(l)$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	2
$\widehat{\alpha}(1,l)$	1	$\frac{1}{2}$	1	1
$\widehat{\alpha}(2,l)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$\widehat{\alpha}(3,l)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$\widehat{\mathrm{j}}(1,l)$	1	4	2	3
$\widehat{\mathrm{j}}(2,l)$	5	5	4	4
$\widehat{\mathrm{j}}(3,l)$	6	6	6	5

Table: $\widehat{\alpha}$ and $\widehat{\beta}$ for $\mathbb{P}_2$ polynomials in two dimensions.

Hierarchical structure $\mathbb{P}_3$ in 2D



$l \in \widehat{\mathcal{L}}$	1	2	3	4	5	6	7	8	9
$\widehat{\beta}(l)$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
$\widehat{\alpha}(1, l)$	1	y	x	y	1	x	y	x	1
$\widehat{\alpha}(2, l)$	$\frac{1}{3}$	y	x	y	$\frac{1}{3}$	x	y	x	$\frac{1}{3}$
$\widehat{\alpha}(3, l)$	$\frac{1}{3}$	z	$\frac{1}{3} - z$	z	$\frac{1}{3}$	$\frac{1}{3} - z$	z	$\frac{1}{3} - z$	$\frac{1}{3}$
$\widehat{\mathrm{j}}(1, l)$	1	6	8	4	2	6	5	4	3
$\widehat{\mathrm{j}}(2, l)$	6	8	9	9	4	7	7	5	5
$\widehat{\mathrm{j}}(3, l)$	8	10	10	10	9	10	10	10	7

Table: $\widehat{\alpha}$ and $\widehat{\beta}$ for $\mathbb{P}_3$ polynomials in two dimensions (uniform lattice). One parameter family $y \in [0, \frac{2}{3}]$, $x = \frac{2}{3} - y$ and $z = \frac{1}{3}(\frac{10}{9} - y)$.

- Define

$$\mathbf{c}_{ij}^{\mathsf{L}} := \sum_{K\in\mathcal{T}(i)} m_i^K \sum_{T\in\mathcal{T}_K^s} \alpha_i^T \frac{1}{|T|}\int_T \nabla\varphi_j^{\mathsf{Lag,L}}\,\mathrm{d}x.$$

- Low-order update

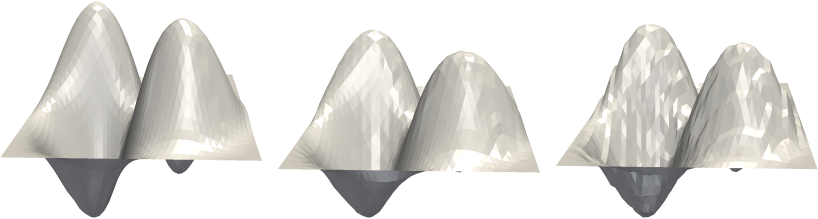
$$m_i\frac{\mathbf{U}_i^{\mathsf{L},n+1}-\mathbf{U}_i^n}{\Delta t} = m_i\mathbf{S}(\mathbf{U}_i^n) - \sum_{j\in\mathcal{I}^{\mathsf{L}}(i)} \mathbb{f}(\mathbf{U}_j^n)\mathbf{c}_{ij}^{\mathsf{L}} + \sum_{j\in\mathcal{I}^{\mathsf{L}}(i)} d_{ij}^{\mathsf{L},n}(\mathbf{U}_j^n-\mathbf{U}_i^n).$$

Theorem (Conservation)

If $\mathbf{S}\equiv\mathbf{0}$, the scheme has the following conservation property:

$$\int_D \mathbf{u}_h^{\mathsf{L},n+1}\,\mathrm{d}x = \int_D \mathbf{u}_h^n\,\mathrm{d}x + \Delta t\int_D \nabla\cdot(\Upsilon_h^{\mathsf{H}}(\mathbb{f}_h(\mathbf{u}_h^n))\,\mathrm{d}x.$$

Rk: Here $\Upsilon_h \equiv \Pi_h^{\mathsf{Lag}}$ if high-order FE are Lagrange.



$\mathbb{P}_1$, 1927 nodes, $\mathbb{P}_2$ 1945 nodes, $\mathbb{P}_3$ 1888 nodes

- If high-order FE is not Lagrange, make change of basis

$$\mathbf{U}^{\mathrm{Lag},n}_{\mathrm{j}(m,K)} := \mathbf{u}^n_h(\mathbf{a}_{\mathrm{j}(m,K)}) = \sum_{l\in\widehat{\mathcal{N}}} b_{lm}\mathbf{U}^n_{\mathrm{j}(l,K)}.$$

- Define Lagrange interpolant of flux

$$\Pi^{\mathrm{Lag}}_h(\mathbb{f}(\mathbf{u}^n_h)) := \sum_{i\in\mathcal{V}} \mathbb{f}(\mathbf{U}^{\mathrm{Lag},n}_{\mathrm{j}(m,K)})\varphi^{\mathrm{Lag}}_i.$$

- Define

$$\mathbf{c}_{ij} := \int_D (\nabla\varphi^{\mathrm{Lag}}_j)\psi_i\,\mathrm{d}x,$$

- High-order update

$$m_i\frac{\mathbf{U}^{\mathrm{H},n+1}_i - \mathbf{U}^n_i}{\Delta t} = -\sum_{j\in\mathcal{I}(i)} \mathbb{f}(\mathbf{U}^{\mathrm{Lag},n}_j)\mathbf{c}_{ij} + \sum_{j\in\mathcal{I}^{\mathrm{L}}(i)} d^{\mathrm{H},n}_{ij}(\mathbf{U}^n_j - \mathbf{U}^n_i).$$

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High-order scheme

- If high-order FE is not Lagrange, make change of basis

$$\mathbf{U}^{\text{Lag},n}_{\mathrm{j}(m,K)} := \mathbf{u}_h^n(\mathbf{a}_{\mathrm{j}(m,K)}) = \sum_{l\in\widehat{\mathcal{N}}} b_{lm}\mathbf{U}^n_{\mathrm{j}(l,K)}.$$

- Define Lagrange interpolant of flux

$$\Pi_h^{\text{Lag}}(\mathbb{f}(\mathbf{u}_h^n)) := \sum_{i\in\mathcal{V}} \mathbb{f}(\mathbf{U}^{\text{Lag},n}_{\mathrm{j}(m,K)})\varphi_i^{\text{Lag}}.$$

- Define

$$\mathbf{c}_{ij} := \int_D (\nabla\varphi_j^{\text{Lag}})\psi_i \,\mathrm{d}x,$$

- High-order update

$$m_i\frac{\mathbf{U}_i^{\mathrm{H},n+1}-\mathbf{U}_i^n}{\Delta t} = -\sum_{j\in\mathcal{I}(i)} \mathbb{f}(\mathbf{U}_j^{\text{Lag},n})\mathbf{c}_{ij} + \sum_{j\in\mathcal{I}^{\mathrm{L}}(i)} d_{ij}^{\mathrm{H},n}(\mathbf{U}_j^n-\mathbf{U}_i^n).$$

High-order scheme

- If high-order FE is not Lagrange, make change of basis

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- High-order update

$$m_i\frac{\mathbf{U}^{\text{H},n+1}_i - \mathbf{U}^n_i}{\Delta t} = -\sum_{j\in\mathcal{I}(i)} \mathbb{f}(\mathbf{U}^{\text{Lag},n}_j)\mathbf{c}_{ij} + \sum_{j\in\mathcal{I}^{\text{L}}(i)} d^{\text{H},n}_{ij}(\mathbf{U}^n_j - \mathbf{U}^n_i).$$

- Compute $m_i\mathbf{U}_i^{\mathrm{H},n+1} - m_i\mathbf{U}_i^{\mathrm{L},n+1}$

$$m_i\mathbf{U}_i^{\mathrm{H},n+1} = m_i\mathbf{U}_i^{\mathrm{L},n+1} - \sum_{K\in\mathcal{K}(i)} \mathbf{B}_{K,i} + \sum_{j\in\mathcal{I}^{\mathrm{L}}(i)} \mathbf{A}_{ij}^n$$

- with dof to dof viscosity contribution

$$\mathbf{A}_{ij}^n := \Delta t(d_{ij}^{\mathrm{H},n} - d_{ij}^{\mathrm{L},n})(\mathbf{U}_j^n - \mathbf{U}_i^n)$$

- and dof to cell flux contribution

$$\mathbf{B}_{i,K}^n := \Delta t \int_K \nabla\cdot(\Pi_h^{\mathrm{Lag}}(\mathbb{f}(\mathbf{u}_h^n)))\psi_i \,\mathrm{d}x - \Delta t m_i^K \sum_{T\in\mathcal{T}_{K,i}^{\mathrm{s}}} \frac{\alpha_i^T}{|T|} \int_T \nabla\cdot(\Upsilon_h^{\mathrm{L}}(\mathbb{f}_h(\mathbf{u}_h^n)))\,\mathrm{d}x.$$

Lemma

Let $\ell_K \in [0,1]$ for all $K \in \mathcal{T}_h$. Let $\ell_{ij} \in [0,1]$ for all $i \in \mathcal{V}$, $j \in \mathcal{I}^{\mathsf{L}}(j)$ with the assumption that $\ell_{ij} = \ell_{ji}$. Let $\mathbf{U}_i^{n+1}$ be defined by

$$m_i \mathbf{U}_i^{n+1} = m_i \mathbf{U}_i^{\mathsf{L},n+1} + \sum_{j \in \mathcal{I}^{\mathsf{L}}(i)} \ell_{ij} \mathbf{A}_{ij}^n + \sum_{K \in \mathcal{T}(i)} \ell_K \mathbf{B}_{i,K}^n.$$

Assume that either

- *the high-order finite element $(\widehat{K}, \widehat{P}^{\mathsf{H}}, \widehat{\Sigma}^{\mathsf{H}})$ is a Lagrange finite element*
- *or the flux $\mathbb{f}$ is linear.*

Then $\mathbf{u}_h^{n+1}$ and $\mathbf{u}_h^{\mathsf{L},n+1}$ carry the same mass, i.e., $\int_D \mathbf{u}_h^{n+1}\,\mathrm{d}x = \int_D \mathbf{u}_h^{\mathsf{L},n+1}\,\mathrm{d}x$.

Claim

Convex limiting can be used to compute limiting coefficients ℓ_{ij} and ℓ_K.
$\Longrightarrow$ High-order solution is conservative and invariant-domain preserving.

Question

Question is open with Bersnstein FE if flux is nonlinear.

The end

The end