How to tie and stabilize vortex knot and link?

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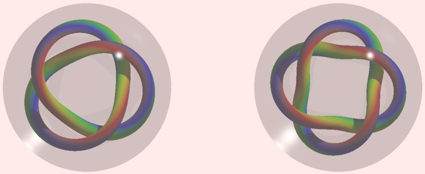
Vortex atom model

W. H. Thomson, “On Vortex Motion,” Trans. R. Soc. Edin., **25**, 217 (1869).

Vortex atom model: knotted or linked vortex in the aether

Aether fluid

atom: knotted or linked vortex



Kelvin’s circulation theorem: knotted or linked vortex lines are conserved in inviscid and barotropic fluid

Stability of knotted or linked vortex

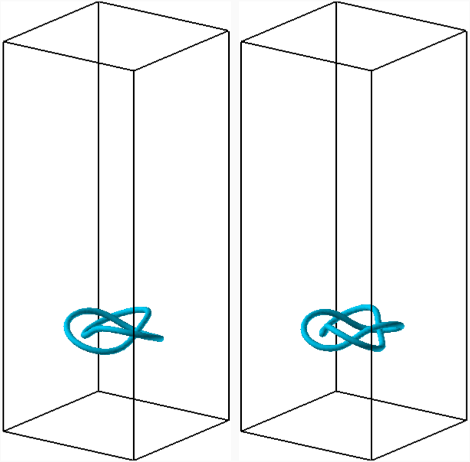
D. Klechkner and W. T. M. Irvine, Nat. Phys. **9**, 253 (2013).

Knotted or linked vortex is unstable in viscous fluid

Vortex ring

Vortex trefoil

Vortex reconnection destabilize knotted or linked vortex

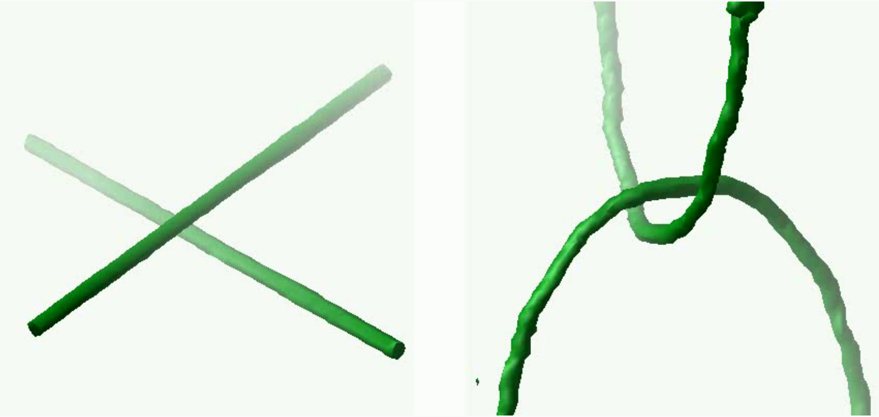


- D. Proment, M. Onorato, and C. F. Barenghi, Phys. Rev. E **85**, 036306 (2012).
- P. C. di Leoni, P. D. Mininni, M. E. Brachet, Phys. Rev. A **94**, 043605 (2016).
- D. Klechkner, L. H. Kauffman, and W. T. M. Irvine, Nat. Phys. **12**, 650 (2016).

Non-Abelian vortex in spin-2 spinor BEC

M. Kobayashi et. al., Phys. Rev. Lett. **103**, 115301 (2009).

Reconnection is forbidden for vortices with non-commutative topological charges



We expect stabilized knot or link when charges are non-commutative for all crossinngs

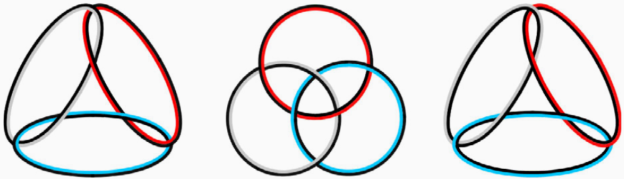
Is it possible to distribute topological charges for knotted or linked vortex?

T. Annala et. al., Comm. Phys. **5**, 309 (2022).

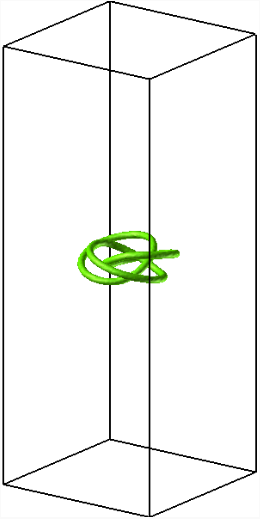
Example: quaternion group

$\mathbb{Q}_8 = \{\pm 1, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$

link with 3-rings by $i\sigma_{x,y,z}$



unstable (simulation)⇒



Can we obtain stable non-Abelian knotted or linked vortex?

Non-Abelian vortex is possible for spin ≥ 2 spinor BEC

Mean-field Hamiltonian for spin-2 snpinor BEC

$$\mathcal{H} = \int d\boldsymbol{x} \left[-\frac{\hbar^2}{2m} \hat{\psi}_s^\dagger \nabla^2 \hat{\psi}_s + \sum_{S=0}^{2s} \frac{g_S}{2} \sum_{S_z=-S}^{S} C_{2s_1 2s_2}^{SS_z} C_{2s_3 2s_4}^{SS_z} \hat{\psi}_{s_1}^\dagger \hat{\psi}_{s_2}^\dagger \hat{\psi}_{s_3} \hat{\psi}_{s_4} \right]$$

$$\xrightarrow{\text{mean-field}} \int d\boldsymbol{x} \left[\frac{\hbar^2}{2m} (\nabla \psi_s^*)(\nabla \psi_s) + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} \boldsymbol{S}^2 + \frac{c_2}{2} |A_{20}|^2 \right]$$

Order parameter and ground state

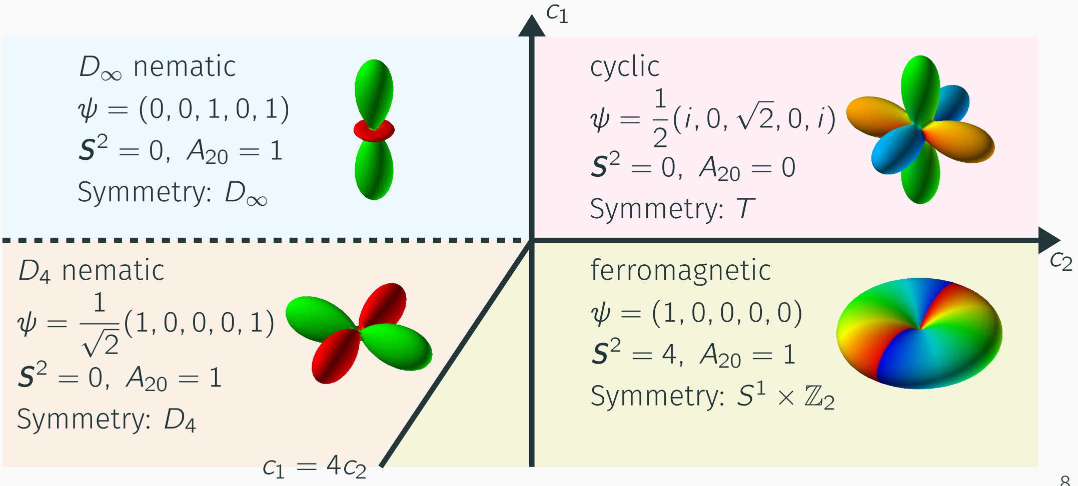
$$c_0 = \frac{4g_2 + 3g_4}{7} \qquad c_1 = \frac{g_4 - g_2}{7} \qquad c_2 = \frac{7g_0 - 10g_2 + 3g_4}{35}$$

$$\psi = \begin{pmatrix} \psi_2 & \psi_1 & \psi_0 & \psi_{-1} & \psi_{-2} \end{pmatrix}^T$$

$$\rho = \psi_s^* \psi_s \qquad \boldsymbol{S} = \psi_s^* \hat{\boldsymbol{S}}_{s\sigma} \psi_\sigma \qquad A_{20} = (-1)^s \psi_s \psi_{-s}$$

Ground-state phase diagram

$$\mathcal{H} = \int dx \left[\frac{\hbar^2}{2m} (\nabla \psi_s^*)(\nabla \psi_s) + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{S}^2 + \frac{c_2}{2}|A_{20}|^2 \right]$$



Ground-state phase diagram

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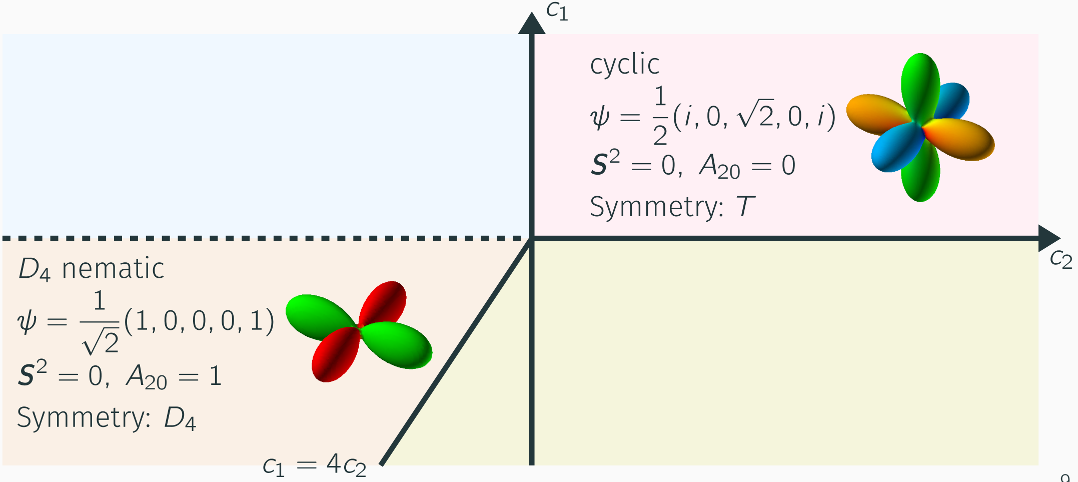

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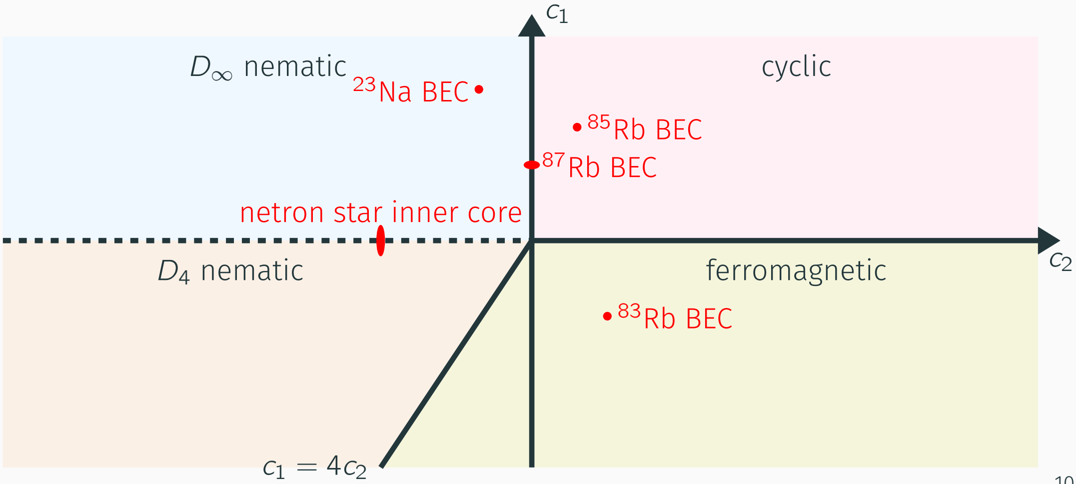

Classification of vortex

Fundamental group of order parameter manifold

$$\pi_1\left(\frac{U(1)\times SO(3)}{S}\right) \cong \pi_1\left(\frac{U(1)\times SU(2)}{S^*}\right) \cong \mathbb{Z}\times_h S^* \qquad \begin{cases}\mathbb{Z} & \text{phase part}\\ S^* & \text{spin part}\end{cases}$$

S: symmetry of the ground state

S^*: double cover of the partial group S of $SO(3)$ $\Rightarrow \dfrac{SO(3)}{S} \simeq \dfrac{SU(2)}{S^*}$

cyclic phase

tetrahedral symmetry: $\pi_1 \cong \mathbb{Z}\times_h T^*$

D_4 nematic phase

4th dihedral symmetry: $\pi_1 \cong \mathbb{Z}\times_h D_4^*$

Classification of vortex

Topological charge (n, s) of vortex is a pair of phase part n and spin part s.

cyclic phase: $\mathbb{Z} \times_h T^*$

classification by conjugacy class

	vacuum / stable / unstable
(I)	$(0, 1)$
(II)	$(0, -1)$
(III)	$(0, \pm i\sigma_{x,y,z})$
(IV)	$(1/3, C_3), (1/3, -i\sigma_{x,y,z}C_3)$
(V)	$(1/3, -C_3), (1/3, i\sigma_{x,y,z}C_3)$
(VI)	$(-1/3, C_3^2), (-1/3, i\sigma_{x,y,z}C_3^2)$
(VII)	$(-1/3, -C_3^2), (-1/3, -i\sigma_{x,y,z}C_3^2)$

$$C_3 \equiv (1 + i\sigma_x + i\sigma_y + i\sigma_z)/2$$

D_4 nematic phase: $\mathbb{Z} \times_h D_4^*$

classification by conjugacy class

	vacuum / stable / unstable
(I)	$(0, 1)$
(II)	$(0, -1)$
(III)	$(0, \pm i\sigma_{x,y})$
(IV)	$(0, \pm i\sigma_z)$
(V)	$(\pm 1/2, C_4), (\pm 1/2, -C_4^3)$
(VI)	$(\pm 1/2, -C_4), (\pm 1/2, C_4^3)$
(VII)	$(\pm 1/2, \pm i\sigma_x C_4), (\pm 1/2, \pm i\sigma_x C_4^3)$

$$C_4 \equiv (1 + i\sigma_z)/\sqrt{2}$$

Topological charge (n, s) of vortex is a pair of phase part n and spin part s.

cyclic phase: $\mathbb{Z} \times_h T^*$

classification by conjugacy class

spin vortex   hydrodynamic vortex

(I)	$(0, 1)$
(II)	$(0, -1)$
(III)	$(0, \pm i\sigma_{x,y,z})$
(IV)	$(1/3, C_3)$, $(1/3, -i\sigma_{x,y,z}C_3)$
(V)	$(1/3, -C_3)$, $(1/3, i\sigma_{x,y,z}C_3)$
(VI)	$(-1/3, C_3^2)$, $(-1/3, i\sigma_{x,y,z}C_3^2)$
(VII)	$(-1/3, -C_3^2)$, $(-1/3, -i\sigma_{x,y,z}C_3^2)$

$$C_3 \equiv (1 + i\sigma_x + i\sigma_y + i\sigma_z)/2$$

D_4 nematic phase: $\mathbb{Z} \times_h D_4^*$

classification by conjugacy class

spin vortex   hydrodynamic vortex

(I)	$(0, 1)$
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$$C_4 \equiv (1 + i\sigma_z)/\sqrt{2}$$

Typical form

cyclic phase: $\mathbb{Z} \times_h T^*$

classification by conjugacy class

spin vortex	hydrodynamic vortex
(III)	$(0, \pm i\sigma_{x,y,z})$ $\Rightarrow \frac{1}{2}(ie^{i\theta}, 0, \sqrt{2}, 0, ie^{-i\theta})$
(IV)	$(1/3, C_3), (1/3, -i\sigma_{x,y,z}C_3)$ $\Rightarrow \frac{1}{\sqrt{3}}(ie^{i\theta}, 0, 0, \sqrt{2}, 0)$
(VII)	$(-1/3, -C_3^2), (-1/3, -i\sigma_{x,y,z}C_3^2)$ $\Rightarrow \frac{1}{\sqrt{3}}(ie^{-i\theta}, 0, 0, \sqrt{2}, 0)$

D_4 nematic phase: $\mathbb{Z} \times_h D_4^*$

classification by conjugacy class

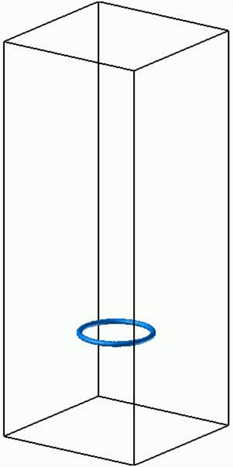
spin vortex	hydrodynamic vortex
(III)	$(0, \pm i\sigma_{x,y})$ $\Rightarrow \frac{1}{\sqrt{2}}(ie^{i\theta}, 0, 0, 0, ie^{-i\theta})$
(V)	$(\pm 1/2, C_4), (\pm 1/2, -C_4^3)$ $\Rightarrow \frac{1}{\sqrt{2}}(e^{i\theta}, 0, 0, 0, 1)$
(VII)	$(\pm 1/2, \pm i\sigma_x C_4), (\pm 1/2, \pm i\sigma_x C_4^3)$ $\Rightarrow \frac{1}{\sqrt{2}}(0, e^{i\theta}, 0, 1, 0)$

Hydrodynamic vortex and spin vortex $\left(\text{dynamics: GP equation } i\hbar\frac{\partial \psi_s}{\partial t} = \frac{\delta \mathcal{H}}{\delta \psi_s^*}\right)$

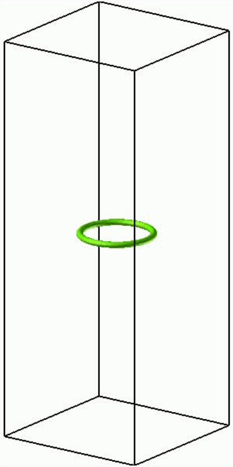
hydrodynamic vortex ring ⇒ stable (self induction)

spin vortex ⇒ unstable (shrink)

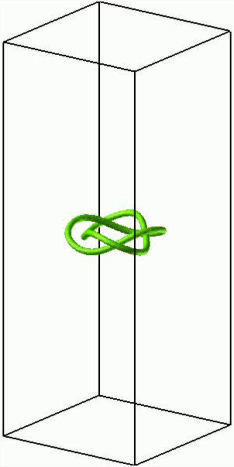
hydrodynamic vortex



spin vortex



knotted spin vortex



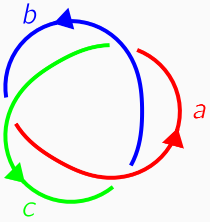
4 elements for stable knotted or linked vortex

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

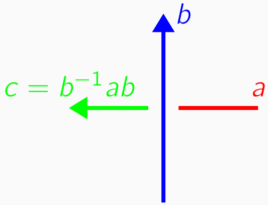
Mutually non-commutativity

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

trefoil example

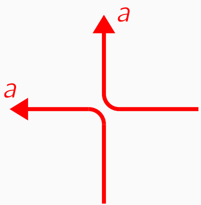


crossing condition



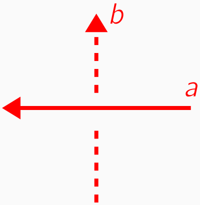
$a^{-1}bcb^{-1} = 1$

reconnection



$a = b$

passing

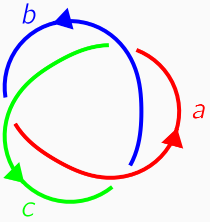


$a \neq b,\ ab = ba$

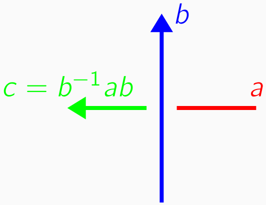
Mutually non-commutativity

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

trefoil example

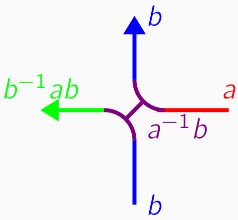


crossing condition



$$a^{-1}bcb^{-1} = 1$$

rung vortex

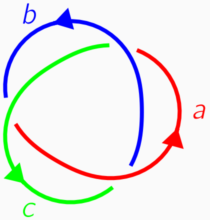


$$ab \neq ba$$

Mutually non-commutativity

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

trefoil example



Topological charges for all arcs are non-commutative
(At least all crossing points)

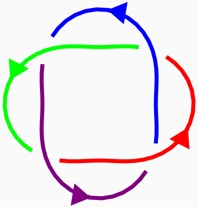
$\Rightarrow$ For given knot or link, whether mutually non-commutativity can be satisfied depends on the group structure for the stable conjugacy class.

Example: All knots are impossible for any Nilpotent group

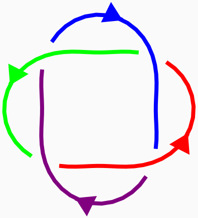
Hydrodynamic orientation

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamic orientation for linked vortex
4. fully positive (or fully negative) braid

All vortex loops advance in the same direction ⇒ stable



Some vortex loops advance in the opposite direction ⇒ unstable

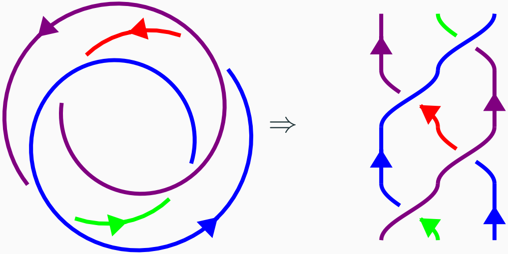


Fully positive (or fully negative) braid

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

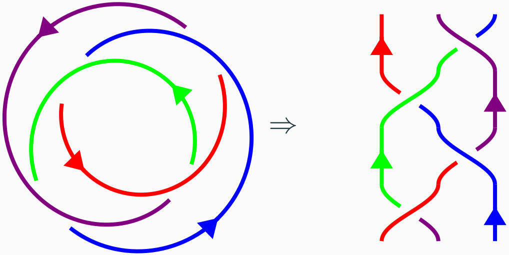
braids with the same sign ⇒ stable

(3, 2)-torus knot (trefoil)



braids with different signs ⇒ unstable

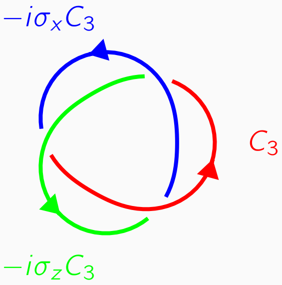
figure-8

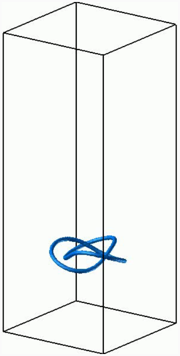


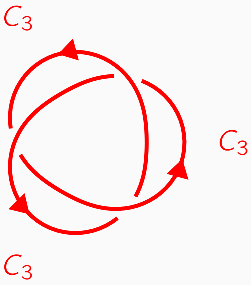
Stable knot in cyclic phase

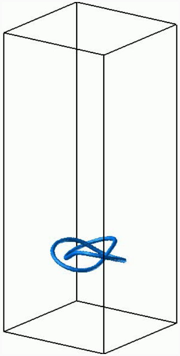
1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

trefoil









1. hydrodynamic vortex
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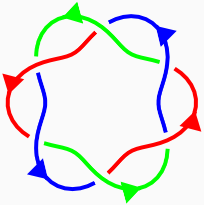
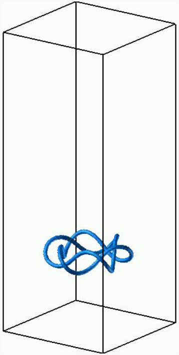
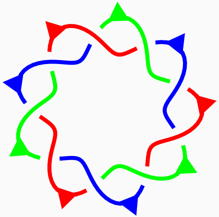
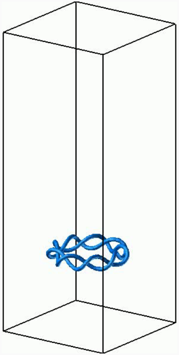
(2, 6)-torus link





(2, 9)-torus knot





Unstable link in cyclic phase

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
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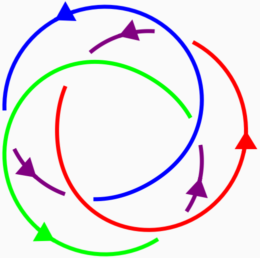
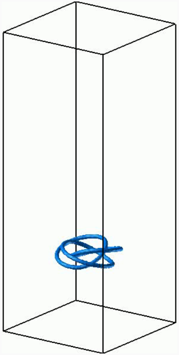
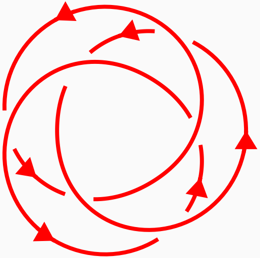
4. fully positive (or fully negative) braid

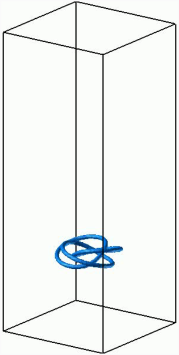
(3, 3)-torus link





(3, 3)-torus link





Unstable link in cyclic phase

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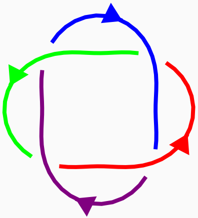
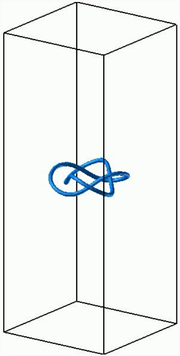
(2, 4)-torus link

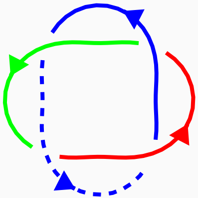


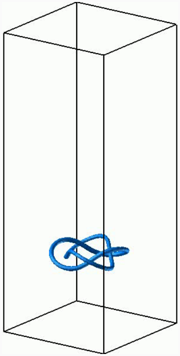


Stable link in nematic phase ($\mathbb{Z} \times_h D_4^*$ is Nilpotent)

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

(2, 4)-torus link

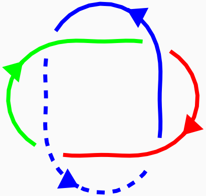


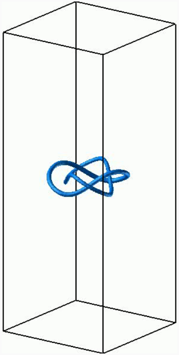


Unstable links in nematic phase ($\mathbb{Z} \times_h D_4^*$ is Nilpotent)

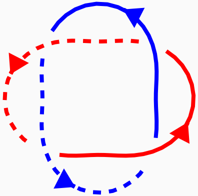
1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

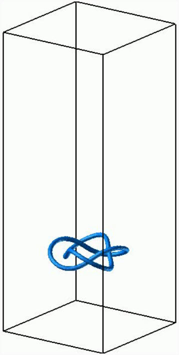
(2, 4)-torus link





(3, 3)-torus link





Summary

Possibility of knotted or linked vortex depends on the group and type of knot or link

Sufficient condition for stability

1. hydrodynamic vortex
2. mutually non-commutative for charges of all arcs
3. hydrodynamically oriented for linked vortex
4. fully positive (or fully negative) braid

Future works

1. necessary condition for stability
2. experimental realization by using spinor BEC instead of aether