Computing the action ground states of nonlinear Schrödinger equation

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Outline

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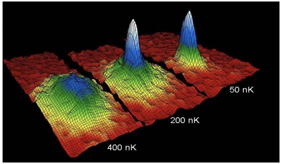
Rotating Nonlinear Schrödinger equation (RNLS)

$$i\partial_t\psi(\mathbf{x},t) = -\frac{1}{2}\Delta\psi + V(\mathbf{x})\psi + \beta|\psi|^{p-1}\psi - \Omega L_z\psi,$$

- $\psi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$ unknown, $\mathbf{x} \in \mathbb{R}^d$, $t > 0$, $i = \sqrt{-1}$
- $V(\mathbf{x}) \geq 0$ external potential, e.g., $V(\mathbf{x}) = \frac{1}{2}\sum_{j=1}^{d}\gamma_j^2 x_j^2$
- $\beta > 0$ (defocusing) or $\beta < 0$ (focusing), $1 < p < \frac{d+2}{d-2}$ (if $d \geq 3$)
- $\Omega \in \mathbb{R}$ the rotational speed, $L_z = -i(x_1\partial_{x_2} - x_2\partial_{x_1})$ if $d \geq 2$; $L_z = 0$ if $d = 1$



Bose-Einstein condensate

Nonlinear optics

Deep water waves

Stationary Equation of RNLS

- Stationary state solutions (solitons):

$$\psi(\mathbf{x},t) = e^{i\omega t}\phi(\mathbf{x}), \quad \omega \in \mathbb{R} \text{ (chemical potential)},$$

$$H(\phi) := -\frac{1}{2}\Delta\phi + V(\mathbf{x})\phi + \beta|\phi|^{p-1}\phi - \Omega L_z\phi + \omega\phi = 0, \quad \phi(\mathbf{x}) \not\equiv 0$$

- Nontrivial solutions of elliptic equation $-\Delta\phi = g(\phi)$ in $\mathbb{R}^d$:
  - Existence & Multiplicity: W. Strauss (1977); H. Berestycki & P.L. Lions (1983) ...
- Identify the relatively stable, physically relevant one [H. Berestycki & P.L. Lions (1983)]:
  - treat ω as a given parameter — action ground state
  - treat ω as a Lagrange multiplier — energy ground state

Definition of Action Ground State: H. Berestycki & P.L. Lions (1983)

- Prescribe a chemical potential $\omega \in \mathbb{R}$. Introduce the action functional:

$$S_{\Omega,\omega}(\phi) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\phi|^2 + V(\mathbf{x})|\phi|^2 + \frac{2\beta}{p+1}|\phi|^{p+1} - \Omega\overline{\phi}L_z\phi + \omega|\phi|^2 \right) d\mathbf{x}$$

- Euler-Lagrange Equation: stationary states $\Longleftrightarrow$ critical points of $S_{\Omega,\omega}(\phi)$,

$$\frac{\delta S_{\Omega,\omega}(\phi)}{\delta\overline{\phi}} = H(\phi) = \left(-\tfrac{1}{2}\Delta + V + \beta|\phi|^{p-1} - \Omega L_z + \omega\right)\phi = 0$$

Action Ground State ϕ_g (minimizes $S_{\Omega,\omega}$ among all the nontrivial solutions)

$$S_{\Omega,\omega}(\phi_g) = \min_{\phi\neq 0,\ H(\phi)=0} S_{\Omega,\omega}(\phi)$$

- $S_{\Omega,\omega}(\phi)$ is not bounded from below in H^1 in focusing case ($\beta < 0$)
- $H(\phi) = 0$ may have infinitely many solutions: H. Berestycki & P.L. Lions (1983) ...

Existing Related Works

- Existence of action ground state in focusing case ($\beta < 0$):
  - Non-rotating, no potential ($\Delta\phi + g(\phi) = 0$): Berestycki & Lions (1983) ...
  - Non-rotating ($\Omega = 0$) with harmonic potential $V(\mathbf{x}) = \frac{1}{2}\sum_{j=1}^{d} \gamma_j^2 x_j^2$: Fukuizumi (2001)
  - Rotating case ($\Omega \in \mathbb{R}$): Ardila & Hajaiej (2021)
  - ...

  Existence & variational characterization in defocusing case ($\beta > 0$) is rarely discussed!

- Numerical methods for computing solitary waves (and multiple nontrivial solutions):
  - Fixed-point iteration type method: Petviashvili & Pokhotelov (1992); ...
  - minimax-type methods: Choi & McKenna (1993); Li & Zhou (2001); ...
  - saddle-point dynamics: E & Zhou (2011); Zhang & Du (2012); Yin, Zhang & Zhang (2019); ...
  - ...

  No specific numerical techniques for computing action ground state ϕ_g of Rotating NLS so far.

Energy ground state

- Energy Ground State: For a given mass $m > 0$,

$$u_g \in \arg \min_{\|u\|_{L^2}^2 = m} E_\Omega(u),$$

where the energy functional [Bao, Wang & Markowich (2005); Bao & Cai (2013)]

$$E_\Omega(u) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + V(\mathbf{x})|u|^2 + \frac{2\beta}{p+1}|u|^{p+1} - \Omega \overline{u} L_z u \right) d\mathbf{x}$$

- Q: Connection between two kinds of ground states?

$$\begin{array}{ccccccc}
\omega & \longrightarrow & \displaystyle\min_{\phi \neq 0, H(\phi)=0} S_{\Omega,\omega}(\phi) & \longrightarrow & \phi_g & \longrightarrow & \|\phi_g\|_{L^2}^2 \\
\vdots & & & & & & \vdots \\
\omega_g & \longleftarrow & u_g & \longleftarrow & \displaystyle\min_{\|u\|_{L^2}^2 = m} E_\Omega(u) & \longleftarrow & m
\end{array}$$

  - Recent results in focusing non-rotating case: Dovetta et al. (2022); Jeanjean & Lu (2022)
  - No results in defocusing case.

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Focusing case: Existence & variational characterization

- **Nehari manifold** $\mathcal{M} = \{\phi \in X\backslash\{0\} : K_{\Omega,\omega}(\phi) = 0\}$, where

$$K_{\Omega,\omega}(\phi) := \int_{\mathbb{R}^d} \overline{\phi} H(\phi) d\mathbf{x} = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\phi|^2 + V|\phi|^2 - \Omega\overline{\phi}L_z\phi + \beta|\phi|^{p+1} + \omega|\phi|^2\right) d\mathbf{x}$$

- Clearly, $\mathcal{M}$ contains all nontrivial solutions of $H(\phi) = 0$

Lemma 1

Let $\beta < 0$, $V(\mathbf{x}) = \frac{1}{2}\sum_{j=1}^{d}\gamma_j^2 x_j^2$, $\gamma_j \geq 0$. If $\omega > -\lambda_0$ (with λ_0 the smallest eigenvalue of $-\frac{1}{2}\Delta + V - \Omega L_z$) and $|\Omega| < \min\{\gamma_1, \gamma_2\}$, then there exists an action ground state ϕ_g and

$$S_{\Omega,\omega}(\phi_g) = \min_{\phi\in\mathcal{M}} S_{\Omega,\omega}(\phi).$$

cf. Berestycki & Lions (1983) ($V = 0, \Omega = 0$); Fukuizumi (2001) ($\Omega = 0$); Ardila & Hajaiej (2021)

Method I: Minimization with Nehari Constraint

- A continuous (projected) gradient flow to solve directly $\min\limits_{\phi \in \mathcal{M}} S_{\Omega,\omega}(\phi)$:

$$\partial_t \phi(\mathbf{x}, t) = -\frac{\delta S_{\Omega,\omega}(\phi)}{\delta \overline{\phi}} + \lambda \frac{\delta K_{\Omega,\omega}(\phi)}{\delta \overline{\phi}} = -H(\phi) + \lambda \frac{\delta K_{\Omega,\omega}(\phi)}{\delta \overline{\phi}}$$

  where $\lambda(t)$ is chosen to preserve the constraint, i.e. $K_{\Omega,\omega}(\phi(\cdot, t)) = 0 \ \forall t > 0$.

- Action-diminishing property: $\dfrac{d}{dt} S_{\Omega,\omega}(\phi) = -2 \, \|\partial_t \phi\|_{L^2}^2$

- Efficient gradient flow schemes and preconditioned optimization methods can be designed [C. Wang (JCP, 2022)], [L-Wen-Yuan-Zhao (JCP 2025)]

- The formulation seems a bit complicated. Can we look for a simplification?

Method II: Minimization with L^{p+1}-normalization

Theorem 2 (L-Yuan-Zhao, SISC, 2023)

Let $\beta < 0$, $\omega > -\lambda_0$, *and* $|\Omega| < \min\{\gamma_1, \gamma_2\}$. *Then*

$$\phi_g \text{ is an action GS} \quad \Longleftrightarrow \quad Q(u_*) = \min_{u \in \mathcal{S}_{p+1}} Q(u)$$

where $u_*(\mathbf{x}) := \phi_g(\mathbf{x})/\|\phi_g\|_{L^{p+1}}$, $\mathcal{S}_{p+1} := \{u \in X \,:\, \|u\|_{L^{p+1}} = 1\}$ *and*

$$Q(u) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + V(\mathbf{x})|u|^2 - \Omega\,\overline{u} L_z u + \omega |u|^2 \right) d\mathbf{x}$$

Moreover, $\phi_g(\mathbf{x}) = \left(\frac{Q(u_*)}{-\beta}\right)^{\frac{1}{p-1}} u_*(\mathbf{x})$.

A continuous normalized gradient flow

- A continuous normalized gradient flow for solving $\min_{u \in \mathcal{S}_{p+1}} Q(u)$:

$$\partial_t u(\mathbf{x},t) = \left(\frac{1}{2}\Delta - V + \Omega L_z - \omega + \lambda(u(\cdot,t))|u|^{p-1}\right)u, \quad u(\cdot,0) = u_0 \in \mathcal{S}_{p+1},$$

$$\lambda(u) = \frac{\int_{\mathbb{R}^d} |u|^{p-1}\overline{u}\left(-\frac{1}{2}\Delta u + (V+\omega)u - \Omega L_z u\right)d\mathbf{x}}{\int_{\mathbb{R}^d} |u|^{2p} d\mathbf{x}}$$

  - Normalization-conservation: $\frac{d}{dt}\|u\|_{L^{p+1}}^{p+1} = 0$
  - Energy diminishing: $\frac{d}{dt}Q(u) = -2\,\|\partial_t u\|_{L^2}^2$

- Drawbacks:
  - strong regularity requirement on u;
  - hard to construct energy-decaying schemes

Gradient flow with asymptotic Lagrange multiplier

- Normalized gradient flow with asymptotic Lagrange multiplier (GFALM) [L-Yuan-Zhao (2023)]:

$$\begin{cases} \partial_t u = \left(\frac{1}{2}\Delta - V + \Omega L_z - \omega + \widetilde{\lambda}(u(\cdot,t_n))|u(\cdot,t_n)|^{p-1}\right)u, \quad t \in [t_n, t_{n+1}), \\ u(\mathbf{x}, t_{n+1}) := u(\mathbf{x}, t_{n+1}^+) = \dfrac{u(\mathbf{x}, t_{n+1}^-)}{\|u(\cdot, t_{n+1}^-)\|_{L^{p+1}}}, \quad n \geq 0, \quad u(\mathbf{x}, 0) = u_0(\mathbf{x}), \end{cases}$$

  where $u_0 \in \mathcal{S}_{p+1}$ and $t_n = n\tau$ $(n \geq 0)$ with $\tau > 0$ a time step.

- $\widetilde{\lambda}$ is an **asymptotic Lagrange multiplier** defined as

$$\widetilde{\lambda}(u) = \frac{Q(u)}{\int_{\mathbb{R}^d} |u|^{p+1} d\mathbf{x}}$$

  In fact, $\widetilde{\lambda}(u(\cdot,t_n)) = Q(u(\cdot,t_n))$ since $u(\cdot,t_n) \in \mathcal{S}_{p+1}$

- A similar framework to GFDN/GFLM for computing L^2-normalized energy ground states: Bao-Du (2004), Bao-Cai (2013), L-Cai (2021), ...

Backward-forward Euler scheme with stabilization

- **GFALM-BF**:

$$\begin{cases} \dfrac{u^{(1)} - u^n}{\tau} = \left(\dfrac{1}{2}\Delta - \alpha_n\right) u^{(1)} + \left(\alpha_n - V + \Omega L_z - \omega + \widetilde{\lambda}^n |u^n|^{p-1}\right) u^n, \\ u^{n+1} = \dfrac{u^{(1)}}{\|u^{(1)}\|_{L^{p+1}}}, \quad n \geq 0, \quad u^0 = u_0 \in \mathcal{S}_{p+1}, \end{cases}$$

  where $\widetilde{\lambda}^n = \widetilde{\lambda}(u^n) = Q(u^n)$ and $\alpha_n \geq 0$ is a stabilization factor.

- **Fixed point**: for $\forall \tau > 0$, $\alpha_n \geq 0$,

$$u^{n+1} = u^n \quad \Longleftrightarrow \quad \left(\widetilde{\lambda}^n, u^n\right) \text{ solves Euler-Lagrange eqn, i.e.,}$$

$$\left(-\tfrac{1}{2}\Delta + V - \Omega L_z + \omega\right) u^n = \widetilde{\lambda}^n |u^n|^{p-1} u^n$$

- Only needs to solve a linear elliptic equation with **constant** coefficients

- In practice, we truncate $\mathbb{R}^d$ to a bounded (box) domain $U \subset \mathbb{R}^d$ and impose the homogeneous Dirichlet BCs or periodic BCs (FFT solver).

Unconditional energy-decaying property

In the following, we consider the GFALM-BF scheme on a suitably large box domain $U \subset \mathbb{R}^d$ with the homogeneous Dirichlet or periodic BCs.

Theorem 3 (L-Yuan-Zhao, SISC 2023; L-Wang-Zhao, arXiv:2602.20820)

Assume that $V \in L^\infty(U)$ *and*

$$\alpha_n \geq \frac{1}{2}\max\left\{0, V(\mathbf{x}) + \omega + \frac{|\Omega|^2}{2}(x_1^2 + x_2^2)\right\} + \frac{1}{2}.$$

Then, GFALM-BF scheme has the unconditionally energy-decaying property: for $\forall \tau > 0$,

$$Q(u^{n+1}) - Q(u^n) \leq -\left(\frac{1}{2}\|u^{n+1} - u^n\|_{H^1}^2 + \frac{2}{\tau}\|u^{n+1} - u^n\|_{L^2}^2\right) \Big/ \|u^{(1)}\|_{L^{p+1}}^2.$$

Global convergence of GFALM

Theorem 4 (L-Wang-Zhao, arXiv:2602.20820)

Suppose that the above Assumptions hold. Let $u^0 \in H^1(U) \cap \mathcal{S}_{p+1}$. *Then, for every fixed* $\tau > 0$:

1. *the solution* $\{u^n\}_{n=0}^{\infty}$ *of GFALM-BF has a strongly convergent subsequence* $\{u^{n_j}\}_{j=0}^{\infty}$ *in* $H^1(U)$;
2. *every accumulation point* $u^\star \in \mathcal{S}_{p+1}$ *of* $\{u^n\}_{n=0}^{\infty}$ *in* $H^1(U)$ *satisfies the Euler-Lagrange equation:*

$$\langle \mathcal{A}u^\star, v\rangle = \lambda^\star \left\langle |u^\star|^{p-1}u^\star, v\right\rangle, \quad \forall v \in H^1(U),$$

with the Lagrange multiplier $\lambda^\star := \widetilde{\lambda}(u^\star) = Q(u^\star) = \inf_{n\geq 0} Q(u^n)$. *Here* $\mathcal{A}u = \frac{\delta Q}{\delta \overline{u}}$.

Local exponential convergence rate

Theorem 5 (L-Wang-Zhao, arXiv:2602.20820)

Suppose that the above basic Assumptions hold, and let u_g be a non-degenerate *action GS. Then, there exists a $\delta_0 > 0$ such that, for any initial guess $u^0 \in U_{\delta_0}(u_g) \cap \mathcal{S}_{p+1}$ and any $\tau > 0$, the GFALM-BF solution $\{u^n\}$ converges to u_g at the exponential rate:*

$$\|u^n - u_g\|_{H^1} \leq C\, \mathrm{e}^{-an\tau/(1+\tau)}, \qquad \forall\, n \in \mathbb{N},$$

where $C, a > 0$ are constants independent of n and τ.

Convergence to discrete GS

- Fourier pseudospectral discretization:

$$u_{g,h} \in \underset{\|u_h\|_{\ell^{p+1}}=1}{\operatorname{argmin}} Q_h(u_h), \quad Q_h(u_h) := \frac{1}{2}\left\langle(-D_{xx}^F)u_h, u_h\right\rangle_h + h\sum_{j=0}^{M-1}(V_j+\omega)\,|u_j|^2 .$$

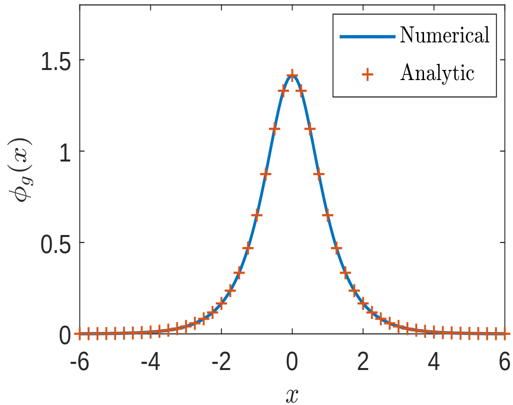
Theorem 6 (L-Wang-Zhao, arXiv:2602.20820)

For any fixed h, assume that the discrete GS $u_{g,h}$ satisfies the non-degeneracy condition.
Then, there exists a $\delta_0 > 0$ such that, for any initial data $u_h^0 \in U_{\delta_0}(u_{g,h}) \cap \mathcal{S}_{h,p+1}$ and any $\tau > 0$, the fully discrete GFALM scheme converges at the rate:

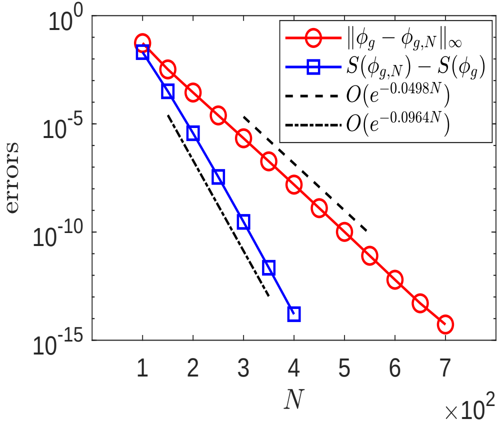
$$\|u_h^n - u_{g,h}\|_{1,h} \le C\,\mathrm{e}^{-an\tau/(1+\tau)}, \qquad \forall\, n \in \mathbb{N}.$$

Spatial spectral accuracy of GFALM-BF

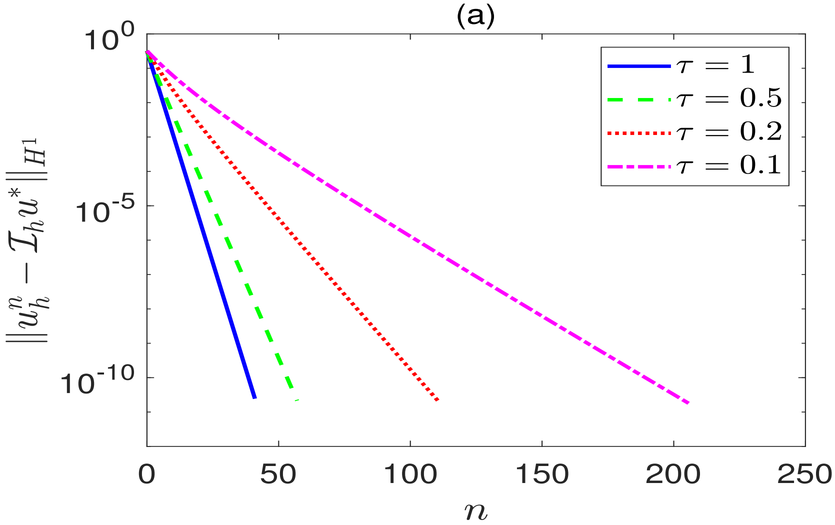
$$\frac{1}{2}\partial_{xx}\phi(x) + |\phi(x)|^2\phi(x) = \phi(x), \quad \phi_g(x) = \sqrt{2}\,\mathrm{sech}\left(\sqrt{2}x\right), \quad x \in \mathbb{R}$$



(a) Profiles of the numerical solution $\phi_{g,N}$ with $N = 2^{10}$ and the analytic solution ϕ_g



(b) Error of the numerical solution with respect to N


(a)
10^0
10^{-5}
10^{-10}
$\|u_h^n - \mathcal{I}_h u^*\|_{H^1}$
0
50
100
150
200
250
n
$\tau = 1$
$\tau = 0.5$
$\tau = 0.2$
$\tau = 0.1$

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Existence & variational characterization in defocusing case

Theorem 7 (L-Yuan-Zhao, SISC 2023; L-Wen-Yuan-Zhao, JCP 2025)

Let $\beta > 0$, $\omega < -\lambda_0$. *Assume that* $V(\mathbf{x}) \geq 0$ $(\forall\, \mathbf{x} \in \mathbb{R}^d)$, $\lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = \infty$, *and*

$$|\Omega| < \gamma_* := \sup\left\{\gamma > 0 : \lim_{|\mathbf{x}|\to\infty} \left(V(\mathbf{x}) - \tfrac{\gamma^2}{2}(x_1^2 + x_2^2)\right) = \infty\right\} \text{ (if } d \geq 2\text{)}.$$

Then, $S_{\Omega,\omega}$ *is bounded from below in* X *and there exists* $\phi_g \in X$ *such that*

$$S_{\Omega,\omega}(\phi_g) = \inf_{\phi\in X} S_{\Omega,\omega}(\phi) = \inf_{\phi\in\mathcal{M}} S_{\Omega,\omega}(\phi).$$

Moreover, ϕ_g *is an action GS* $\iff$ $u_* := \phi_g/\|\phi_g\|_{L^{p+1}} \in \arg\min_{u\in\mathcal{S}_{p+1}} Q(u)$.

- Various numerical algorithms can be designed based on each of three minimization formulations [L-Wen-Yuan-Zhao (JCP 2025)]

Unconstrained minimization methods

- A direct gradient flow for unconstrained minimization:

$$\partial_t \phi(\mathbf{x}, t) = -\frac{\delta S_{\Omega,\omega}(\phi)}{\delta \overline{\phi}} = \frac{1}{2}\Delta\phi - V\phi - \beta|\phi|^{p-1}\phi + \Omega L_z \phi - \omega\phi, \ \ t \geq 0,$$

  - **Backward-forward Euler scheme (GF-BF):**

$$\frac{\phi^{n+1} - \phi^n}{\tau} = \frac{1}{2}\Delta\phi^{n+1} - \alpha_n \phi^{n+1} + \left(\alpha_n - V - \omega - \beta|\phi^n|^{p-1} + \Omega L_z\right)\phi^n,$$

  - Unconditionally action-decaying property (for suitably large α_n)
  - Convergence analysis in rotating case: [L-Wang-Yuan-Zhao (arXiv:2605.04485)]

- Preconditioned optimization methods ... [L-Yuan-Zhao (SISC/2023)]; [L-Wen-Yuan-Zhao (JCP/2025)]

Unconditionally decay of action

Theorem 8 (L-Wang-Yuan-Zhao, arXiv:2605.04485)

Let U be a box domain, $V \in L^\infty(U)$, $\beta > 0$ and $\phi^0 \in X$. There exists a constant $C_0 = C_0(U, V, \beta, p, \phi_0)$ s.t., when $\alpha \geq \frac{1}{2} \max\left\{0,\ \operatorname{ess\,sup}_{\mathbf{x}\in U} \left(V(\mathbf{x}) + \omega + 2|\Omega|^2(x_1^2 + x_2^2)\right)\right\} + C_0$, the GF-BF scheme

$$\frac{\phi^{n+1} - \phi^n}{\tau} = \frac{1}{2}\Delta\phi^{n+1} - \alpha\phi^{n+1} + \left(\alpha - V - \omega - \beta|\phi^n|^{p-1} + \Omega L_z\right)\phi^n,$$

possesses the unconditional decaying property for the action functional $S_{\Omega,\omega}$: $\forall \tau > 0$,

$$S_{\Omega,\omega}(\phi^{n+1}) - S_{\Omega,\omega}(\phi^n) \leq -\frac{2}{\tau}\left\|\phi^{n+1} - \phi^n\right\|_{L^2}^2 - \frac{1}{4}\left\|\phi^{n+1} - \phi^n\right\|_{H^1}^2.$$

Based on this property, we can further establish a global convergence and a local exponential convergence rate to the action ground state under a reasonable non-degeneracy condition [L-Wang-Yuan-Zhao (arXiv:2605.04485)].

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Main attention: Relation with energy ground states

- **Energy Ground State** $u_g = u_g(\mathbf{x})$ minimizes the energy [Bao & Cai (2013)]

$$E_\Omega(u) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + V(\mathbf{x})|u|^2 + \frac{2\beta}{p+1}|u|^{p+1} - \Omega \overline{u} L_z u \right) d\mathbf{x}$$

under the constraint $\|u\|_{L^2}^2 = m$ with given mass $m > 0$.

  - The chemical potential ω_g serves as a Lagrange multiplier:

$$\omega_g = -\mu(u_g), \quad \mu(u) := \frac{1}{m} \int_{\mathbb{R}^d} \left(\tfrac{1}{2}|\nabla u|^2 + V(\mathbf{x})|u|^2 + \beta|u|^{p+1} - \Omega \overline{u} L_z u \right) d\mathbf{x}$$

- **Q: Connection between two kinds of ground states?**

$$\begin{array}{ccccccc}
\omega & \longrightarrow & \min\limits_{\phi \neq 0, H(\phi)=0} S_{\Omega,\omega}(\phi) & \longrightarrow & \phi_g & \longrightarrow & \|\phi_g\|_{L^2}^2 \\
\vdots & & & & & & \vdots \\
\omega_g & \longleftarrow & u_g & \longleftarrow & \min\limits_{\|u\|_{L^2}^2 = m} E_\Omega(u) & \longleftarrow & m
\end{array}$$

Equivalence in non-rotating case

In the following, we focus on the defocusing case ($\beta > 0$).

Theorem 9 (L-Wang-Zhao, M3AS, 2025)

Let $\Omega = 0$. Then two kinds of ground states are equivalent:

- *For any $\omega < -\lambda_0$, an action ground state ϕ_ω associated with ω is also an energy ground state with mass $m = \|\phi_\omega\|_{L^2}^2$.*
- *For any $m > 0$, an energy ground state u_g with mass m is also an action ground state associated with $\omega = -\mu(u_g)$ where $\mu(u) := \frac{1}{\|u\|_{L^2}^2} \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + V(\mathbf{x})|u|^2 + \beta|u|^{p+1}\right) d\mathbf{x}$.*

Conditional equivalence in rotating case

- In rotating case ($\Omega \neq 0$), due to the non-uniqueness, we consider

$$\phi_{\Omega,\omega} \in \mathcal{A}_{\Omega,\omega} := \arg\min_{\phi \in X} S_{\Omega,\omega}(\phi), \quad \phi_m^E \in \mathcal{B}_{\Omega,m} := \arg \min_{\|\phi\|_{L^2}^2 = m} E(\phi),$$

Theorem 10 (L-Wang-Zhao, M3AS, 2025)

For any $\omega < -\lambda_0(\Omega)$, the action ground state $\phi_{\Omega,\omega}$ is also an energy ground state with mass $m = \|\phi_{\Omega,\omega}\|_{L^2}^2$, and all the energy ground states ϕ_m^E with precisely this mass m are also action ground states associated with ω. Moreover,

$$\mathcal{A}_{\Omega,\omega} = \bigcup_{m \in \mathcal{N}_{\Omega,\omega}} \mathcal{B}_{\Omega,m}, \quad \textit{where } \mathcal{N}_{\Omega,\omega} := \left\{ \|\phi\|_{L^2}^2 : \phi \in \mathcal{A}_{\Omega,\omega} \right\}.$$

- **Q: Is there $m > 0$ that is not the mass of any action ground state?**

Numerical exploration: Vortices and critical angular velocity

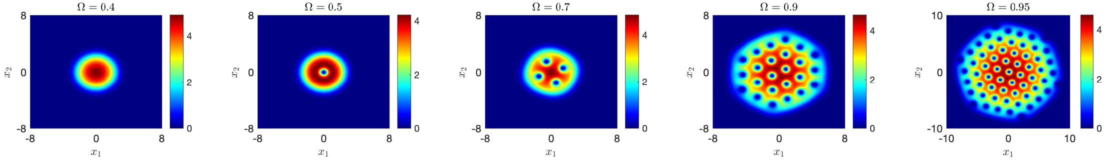


Figure 1: Density profiles of $|\phi_{\Omega,\omega}(x_1, x_2)|^2$ for $\omega = -5$ and different Ω.

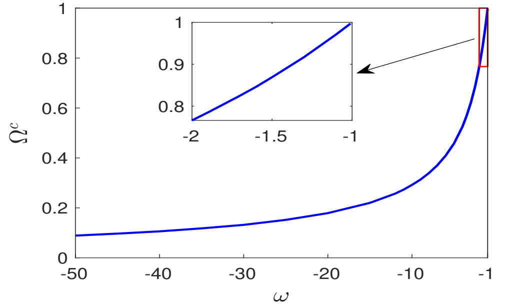
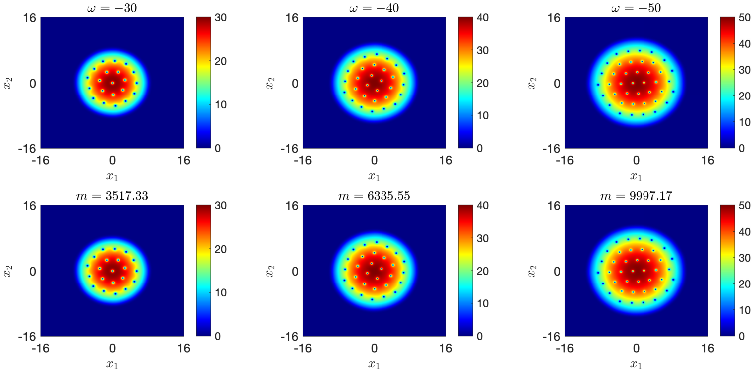


Figure 2: Critical angular velocity Ω^c as a function of ω

Examples of equivalence in rotating case ($\Omega = 0.5$)

Action ground states (1st row) and corresponding energy ground states (2nd row)



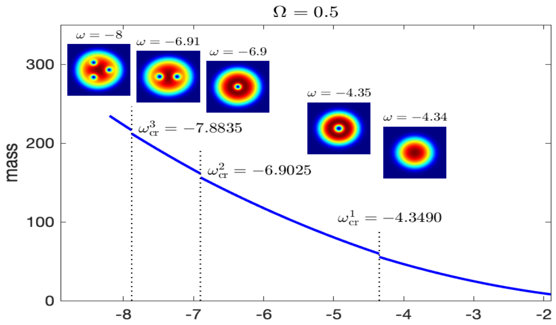
ω	S_g	m	E_g	μ_g	e_ω^{rel}	e_S^{rel}
-30	-34342.56	3517.33	71177.33	30.00	3.21×10^{-10}	7.31×10^{-14}
-40	-82943.09	6335.55	170478.80	40.00	6.59×10^{-10}	4.82×10^{-14}
-50	-163950.62	9997.17	335907.93	50.00	2.21×10^{-10}	2.38×10^{-14}

Numerical evidence of non-equivalence: Jump of mass

- Observation: the mass of the action ground state $\|\phi_{\Omega,\omega}\|_{L^2}^2$ has jump discontinuities at $\omega_{\rm cr}^1 = -4.3490$, $\omega_{\rm cr}^2 = -6.9025$, $\omega_{\rm cr}^3 = -7.8835$, etc.
- No action ground state with a mass value coming from the range

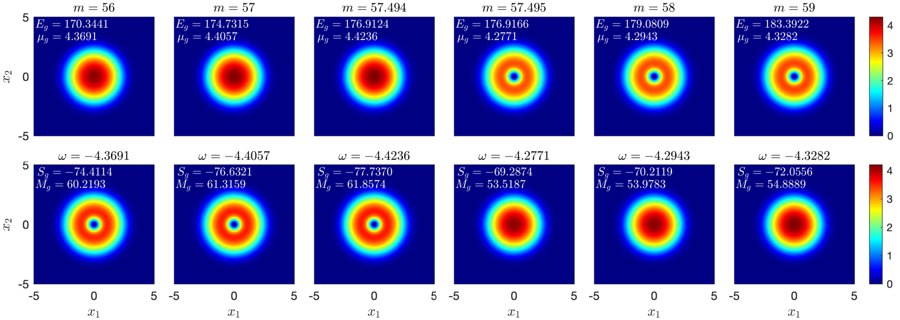
$$(55.4535, 59.6198) \bigcup (156.3416, 161.2327) \bigcup (211.8255, 216.2082) \bigcup \cdots.$$

- $\omega_{\rm cr}^j$ are exactly the transition points where the number of vortices in $\phi_{\Omega,\omega}$.



Numerical evidence: Energy GSs that are not action GSs

- For $m \in (55.4535, 59.6198)$, the energy ground states ϕ_m^E (1st row) are not any action ground state (2nd row), and there exists a critical value $m_{\rm cr}^1 \in (57.494, 57.495)$ s.t.
  - when $m \in (55.4535, m_{\rm cr}^1)$, $\omega = -\mu(\phi_m^E) < \omega_{\rm cr}^1$ and $\|\phi_{\Omega,\omega}\|_{L^2}^2 > 59.6198$;
  - when $m \in (m_{\rm cr}^1, 59.6198)$, $\omega = -\mu(\phi_m^E) > \omega_{\rm cr}^1$ and $\|\phi_{\Omega,\omega}\|_{L^2}^2 < 55.4535$.
- These energy ground states are nontrivial solutions to the stationary RNLS with $\omega = -\mu(\phi_m^E)$ and higher action values, e.g. for $m = 57.494$, $S_{\Omega,\omega}(\phi_m^E) = -77.4181 > S_g = -77.7370$



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Conclusion

In focusing case ($\beta < 0$):

- $\phi_g \iff$ Nehari minimization $\iff$ minimization with L^{p+1}-normalization
- Gradient flow with asymptotic Lagrange multiplier (GFALM) and its convergence

In defocusing case ($\beta > 0$):

- Existence, variational characterization, minimization algorithms:

  $\phi_g \iff$ Nehari minimization $\iff$ L^{p+1}-constrained min $\iff$ unconstrained min.

- Action GS vs. Energy GS:
  - Rotating case: Conditional equivalence & Numerical evidence for non-equivalence
  - Non-rotating case: complete equivalence

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Thank you!