On inviscid instabilities of 2D vortices

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JFM PAPERS

On the inviscid instability of the 2-D Taylor–Green vortex

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JFM PAPERS

On the linear stability of the Lamb–Chaplygin dipole

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- 2D Euler system on a periodic domain

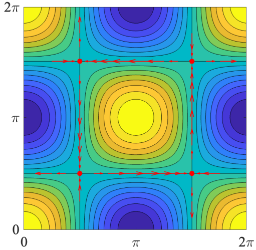
$$\partial_t \omega + (\boldsymbol{u} \cdot \nabla)\omega = 0, \qquad (\boldsymbol{x}, t) \in \mathbb{T}^2 \times (0, T],$$
$$\omega\big|_{t=0} = \omega_0, \qquad \boldsymbol{x} \in \mathbb{T}^2,$$

  where
  - $\omega = \nabla \times \boldsymbol{u}$ is the scalar vorticity field satisfying $\int_{\mathbb{T}^2} \omega \, d\boldsymbol{x} = 0$
  - Biot-Savart law: $\boldsymbol{u} = \nabla^{\perp}\Delta^{-1}\omega, \qquad \nabla^{\perp} = (-\partial_y, \partial_x)$

- An equilibrium solution — the 2D Taylor-Green vortex:

$$\boldsymbol{u}_s = (-\cos(x_1)\sin(x_2), \sin(x_1)\cos(x_2))$$

$$\omega_s = 2\cos(x_1)\cos(x_2)$$



- Linearize the 2D Euler system around a steady solution $(\omega_s, \boldsymbol{u}_s) \Longrightarrow$

$$
\begin{aligned}
\partial_t w &= \mathcal{L} w, \qquad w\big|_{t=0} = w_0, \qquad \text{where} \\
\mathcal{L} w &:= -(\boldsymbol{u}_s \cdot \nabla) w - (\boldsymbol{u} \cdot \nabla)\omega_s \\
&= \left[-\boldsymbol{u}_s \cdot \nabla - \nabla\omega_s \cdot (\nabla^{\perp}\Delta^{-2}) \right] w, \qquad w \in H_0^m(\mathbb{T}^2)
\end{aligned}
$$

- Linear stability of the equilibria determined by the spectrum of the operator $\mathcal{L}$

$$
\begin{aligned}
\sigma(\mathcal{L}) &:= \{\lambda \in \mathbb{C} \;:\; (\mathcal{L} - \lambda) \text{ is not bounded from below in } \mathcal{X}\} \\
&= \sigma_{\text{disc}}(\mathcal{L}) \cup \sigma_{\text{ess}}(\mathcal{L})
\end{aligned}
$$

  - A point $z \in \sigma_{\text{disc}}(\mathcal{L})$ if
    (1) z is an isolated point in $\sigma(\mathcal{L})$;
    (2) z has finite multiplicity, i.e., $\bigcup_{r=1}^{\infty} \operatorname{Ker}(z - \mathcal{L})^r$ is finite dimensional;
    (3) The range of $z - \mathcal{L}$ is closed.
  - Otherwise, $z \in \sigma_{\text{ess}}(\mathcal{L})$

- The set of eigenvalues of $\mathcal{L}$ forms its point spectrum

$$\sigma_{\mathrm{p}}(\mathcal{L}) := \left\{\lambda \in \mathbb{C} \ : \ \exists \phi(\mathbf{x}) \neq 0, \quad \mathcal{L}\phi(\mathbf{x}) = \lambda\phi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{T}^2\right\}$$

- In finite dimensions linear operators can only have point spectrum
- In infinite dimensions the situation is complicated by the presence of the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$, which may also contain eigenvalues of $\mathcal{L}$ that are not in $\sigma_{\mathrm{disc}}(\mathcal{L})$

- For the 2D linear Euler operator $\mathcal{L}$ the essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$ is well understood (Friedlander et al. 1997, Shvydkoy & Latushkin 2003)

$$\sigma_{\text{ess}}(\mathcal{L}; H_0^m) = \{z \in \mathbb{C} : |\Re(z)| \le |m| \, \mu_{\max}\},$$

where

$$\mu_{\max} := \lim_{t \to \infty} \frac{1}{t} \log \sup_{\boldsymbol{x} \in \mathbb{T}^2} ||\nabla \varphi_t(\boldsymbol{x})||$$

is the maximal Lyapunov exponent corresponding to the Lagrangian flow $\varphi_t : \boldsymbol{\xi} \to \boldsymbol{x}(t; \boldsymbol{\xi})$ generated by the steady state via $\partial_t \boldsymbol{x}(t) = \boldsymbol{u}_s(\boldsymbol{x}(t))$

- If $z \in \sigma_{\text{ess}}(\mathcal{L})$, then there exists a sequence of weak-null unit vectors $\{f_n\} \in H_0^m(\mathbb{T}^2)$, approximate eigenvectors, such that

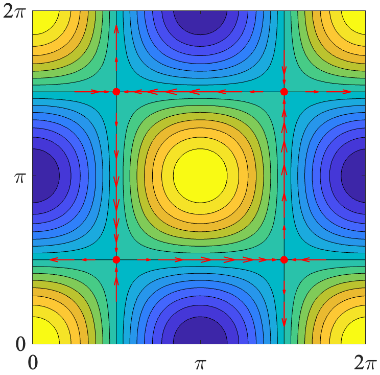
$$\|(\mathcal{L} - z) f_n\|_{H^m} \to 0, \qquad n \to \infty$$

and $\{f_n\}$ does not contain any convergent subsequence

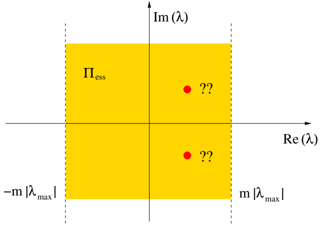
- The the maximal Lyapunov exponent is determined by the hyperbolic stagnation point

$$\mu_{\max} = \max\left[\lambda\left(\boldsymbol{\nabla}\mathbf{u}_s(\mathbf{x}_s)\right)\right]$$

- In the Taylor-Green vortex: $\mu_{\max} = 1$



- No general results available about the discrete spectrum $\sigma_{\text{disc}}(\mathcal{L})$
- Unstable eigenvalues found for shear flows (Drazin & Reid 1981; Friedlander et al. 1997), Kolmogorov flows (Friedlander et al. 1997) and the cellular "cat's-eye" flow (Friedlander et al. 2000).
- The problem remains open for the Taylor-Green vortex
  - Some work in the viscous setting (Sipp & Jacquin 1998; Leblanc & Godeferd 1999; Hattori & Hirota 2023). However, in that case the vortex decays as $\mathcal{O}(e^{-\nu t})$ and $\sigma_{\text{ess}}(\mathcal{L}) = \emptyset$
- QUESTION: does $\mathcal{L}$ have unstable eigenvalues and, if so, do they belong to $\sigma_{\text{disc}}(\mathcal{L})$ or $\sigma_{\text{ess}}(\mathcal{L})$?



Theorem (Lin 2004)

Suppose there exists an exponentially growing solution $e^{\lambda t} w_0$ of the linearized system $\partial_t w = \mathcal{L} w$ with $\Re(\lambda) > 0$ and let $w_0 \in L^2(\mathbb{T}^2)$. Then we have the following

(i) *[regularity of growing modes] $w_0 \in W^{1,p}(\mathbb{T}^2) \bigcap L^q(\mathbb{T}^2)$ for all $1 \le p < p^*$ and $1 \le q < \infty$, where*

$$p^* = \begin{cases} \frac{\mu_{max}}{\mu_{max} - \Re(\lambda)} = \frac{1}{1 - \Re(\lambda)}, & \mu_{max} > \Re(\lambda) \\ \infty, & \mu_{max} \le \Re(\lambda) \end{cases}$$

(ii) *[nonlinear instability] if there exists $\epsilon > 0$, such that for any $\delta > 0$ there is a solution $w^\delta(t)$ of the 2D Euler system corresponding to the initial condition ω_0^δ, such that*

$$\|\omega_0^\delta - \omega_s\|_{L^q} + \|\omega_0^\delta - \omega_s\|_{W^{1,p}} \le \delta,$$

and

$$\sup_{0<t<T_\delta} \|\omega^\delta(t) - \omega_s\|_{H^m} \ge \epsilon,$$

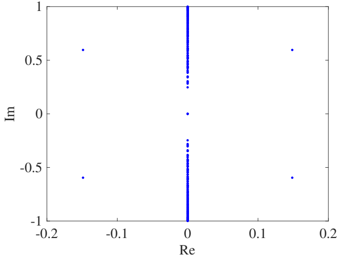
where $T_\delta = O(|\ln(\delta)|)$, $q \in [1, \infty)$, $p = [1, p^*)$ and $m \ge -1$.

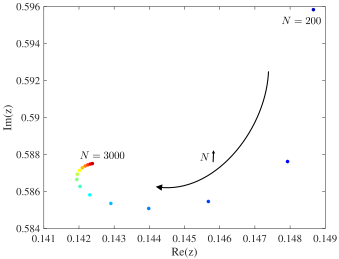
- Numerical solution of the eigenvalue problem $\mathcal{L}\phi(\mathbf{x}) = \lambda\phi(\mathbf{x})$
- Discretize $\mathcal{L}$ using the basis

$$\varphi_{j_1,j_2} = \frac{1}{\sqrt{2}\pi}\cos(j_1 x + j_2 y), \qquad \psi_{j_1,j_2} = -\frac{1}{\sqrt{2}\pi}\sin(j_1 x + j_2 y),$$
$$(j_1, j_2) \in \{(0, j_2), 1 \le j_2 \le N\} \bigcup \{(j_1, j_2), 1 \le j_1 \le N, -N \le j_2 \le N\}$$

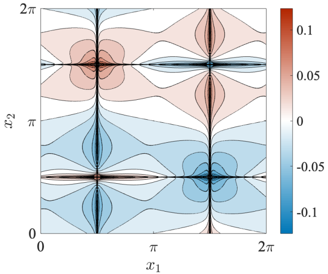
to obtain a sparse algebraic eigenvalue problem $\mathbf{L}\,\phi = \lambda\phi$

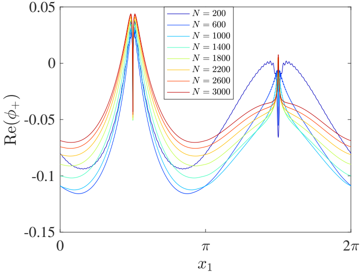
- The spectrum of **L**





- The unstable eigenvalue: $\lambda_+^{3000} = 0.1424 + 0.5875i$
- The corresponding eigenvectors $\Re\left[\phi_+^N\right]$ are L^2 integrable





- From the theorem of Lin (2004):

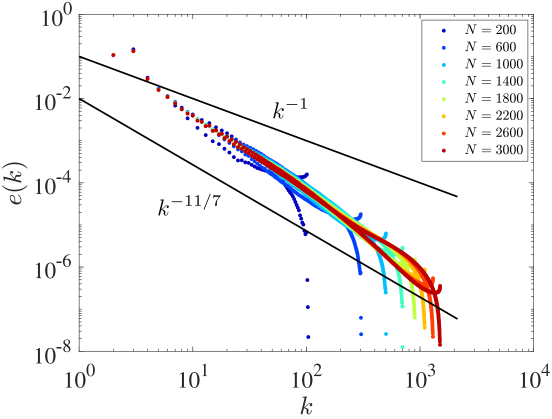
$$\phi_+ \in W^{1,p^*}(\mathbb{T}^2), \qquad \text{where} \qquad p^* = \frac{1}{1 - 0.14} = 1.16$$

- From the Sobolev embedding $W^{1,p^*} \hookrightarrow H^s$ with $1/p^* - 1/2 = 1/2 - s/2$, we have

$$\phi_+ \in H^{0.28}(\mathbb{T}^2)$$

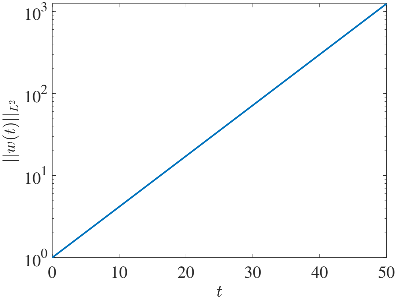
The Fourier spectrum of the eigenfunction ϕ

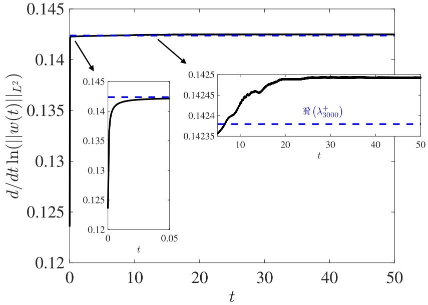
$$e(k) := \frac{1}{2} \sum_{k \le |\boldsymbol{j}| < k+1} |\widehat{\phi}_{\boldsymbol{j}}|^2, \qquad k \in \mathbb{N}$$



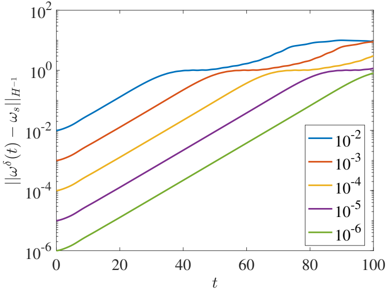
$e(k) = \mathcal{O}\left(k^{-1}\right)$ implies L^2 summability; $e(k) = \mathcal{O}\left(k^{-1.65}\right)$ means $\phi_+ \in H^{0.28}(\mathbb{T}^2)$

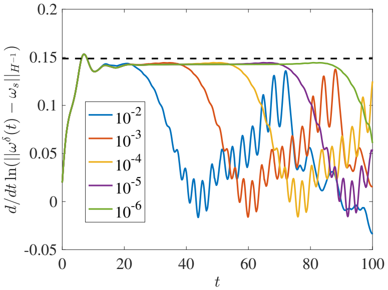
Exponential growth of the perturbation





The actual growth rate is within 0.01% equal $\Re(\lambda_+^{3000})$





The time-dependence of (left) the norm $\|\omega^\delta(t) - \omega_s\|_{H^{-1}}$ and (right) the growth rate $(d/dt)\ln(\|\omega^\delta(t) - \omega_s\|_{H^{-1}})$ in the solution of the nonlinear problem with the initial condition given in terms of the eigenfunction ϕ_+^{200} as $\omega_0 = \omega_s + \delta \cdot \Re\left(\phi_+^{200}\right) / ||\Re\left(\phi_+^{200}\right)||_{H^{-1}}$ with different indicated magnitudes δ.

- The growth rate of the eigenfunction with $\Re(\lambda_+0) \approx 0.1424$ does not saturate the bound on the growth of the semigroup

$$\left\| e^{\mathcal{L}t} w_0 \right\|_{H^m} \le \| w_0 \|_{H^m} e^{\mu_{\max} t}, \qquad \mu_{\max} = 1, \quad m = \pm 1$$

- How to find initial data w_0 realizing this growth?

- Given $T > 0$, define the objective functional $J_m \; : \; H_0^m(\mathbb{T}^2) \to \mathbb{R}$

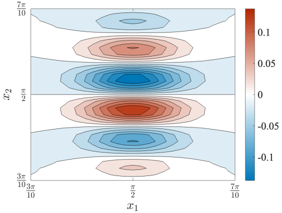
$$J_m(w_0) = \left\| e^{T\mathcal{L}} w_0 \right\|_{H^m}^2, \qquad m = \pm 1$$

Problem (optimization)

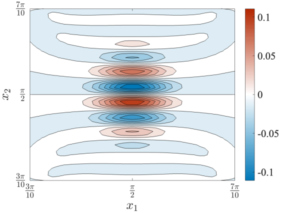
$$\max_{w_0 \in \mathcal{M}} J_m(w_0), \quad \textit{where } m = \pm 1 \textit{ and}$$
$$\mathcal{M} := \{ w_0 \in H_0^m(\mathbb{T}^2) \ : \ \|w_0\|_{H^m} = 1 \}.$$

$$\textit{subject to:} \quad \begin{cases} \partial_t w - \mathcal{L} w = 0, & \textit{in } \mathbb{T}^2 \times (0, T] \\ w = w_0 & \textit{in } \mathbb{T}^2 \textit{ at } t = 0 \\ \textit{Periodic Boundary Conditions} & \end{cases}$$

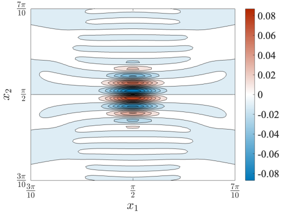
- Problem solved with a discretized gradient flow



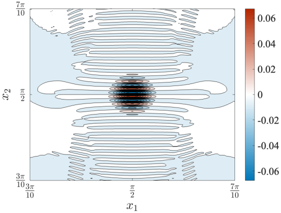
(a) $N = 128$



(b) $N = 256$

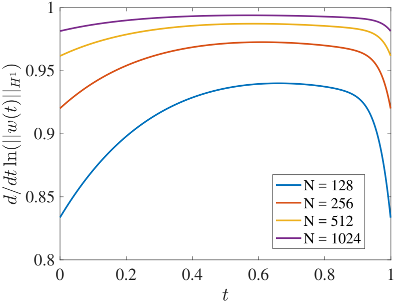


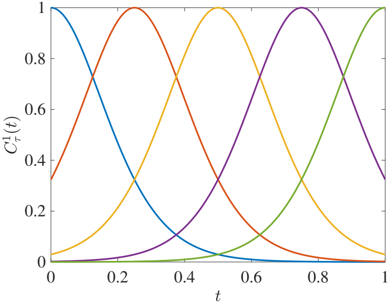
(c) $N = 512$



(d) $N = 1024$

$[m = 1]$ The optimal initial conditions $\widetilde{w}_0^N$ obtained using different resolutions N.

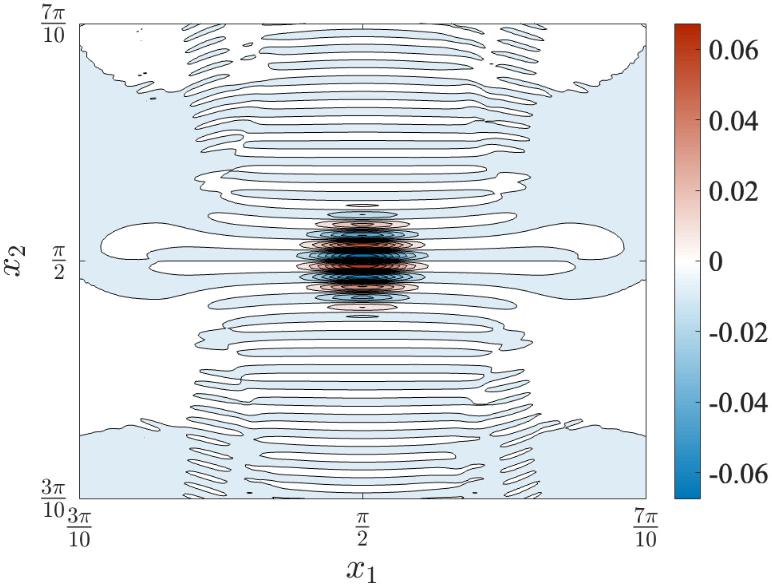




[$m = 1$] (left) The growth rate $(d/dt)\ln(\|w^N(t)\|_{H^1})$ versus t for the optimal initial conditions $\widetilde{w}_0^N$ obtained using increasing spatial resolutions N^2. (right) The time-dependence of the autocorrelation function

$$\mathcal{C}_\tau^1(t) := \frac{\langle w(\tau), w(t)\rangle_{H^1}}{\|w(\tau)\|_{H^1}\|w(t)\|_{H^1}}, \qquad t,\tau \geq 0$$

corresponding to $N = 1024$ and $\tau = 0, 0.25, 0.5, 0.75, 1$. Similar results obtained for $m = -1$.


$\frac{7\pi}{10}$
x_2
$\frac{\pi}{2}$
$\frac{3\pi}{10}$
$\frac{3\pi}{10}$
$\frac{\pi}{2}$
$\frac{7\pi}{10}$
x_1
0.06
0.04
0.02
0
-0.02
-0.04
-0.06

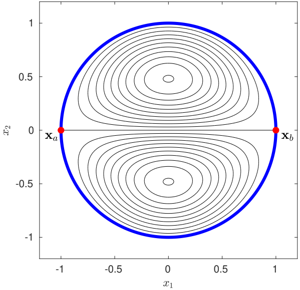
- Consider 2D Euler flow governed by $\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\,\omega = 0$
- Relative equilibria $\{\psi_0, \omega_0\}$ satisfy

$$-\Delta\psi_0 = \omega_0 = F(\psi) \qquad \text{for some } \textit{vorticity} \text{ function } F \;:\; \mathbb{R} \to \mathbb{R}$$

- For the Lamb-Chaplygin dipole (Lamb 1895; Chaplygin 1903):

$$F(\psi) = \begin{cases} -b^2\psi, & \psi > 0 \\ 0, & \text{otherwise} \end{cases}, \qquad \text{where } J_1(b) = 0$$

$$\omega_0(r,\theta) = \begin{cases} \frac{2Ub}{J_0(b)} J_1(br)\sin\theta, & 0 < r \le 1 \\ 0, & r > 1 \end{cases}$$



- The question of stability remains open (Meleshko & van Heijst, 1994)

- The linear stability of the dipole determined by the linearized Euler equations (U is the translation velocity)

$$
\begin{aligned}
\frac{\partial \omega'}{\partial t} &= -\left(\boldsymbol{\nabla}^{\perp}\psi_0 - U\mathbf{e}_1\right)\cdot\boldsymbol{\nabla}\omega' - \boldsymbol{\nabla}^{\perp}\psi_1'\cdot\boldsymbol{\nabla}\omega_0, && \text{in } A_0,\\
\Delta\psi_1' &= -\omega', && \text{in } A_0,\\
\Delta\psi_2' &= 0, && \text{in } \mathbb{R}^2\backslash\overline{A}_0,\\
\psi_1' = \psi_2' &= f', \quad \frac{\partial\psi_1'}{\partial n} = \frac{\partial\psi_2'}{\partial n} && \text{on } \partial A_0,\\
\psi_2' &\to 0, && \text{for } |\mathbf{x}|\to\infty,\\
\omega'(0) &= w', && \text{in } A_0,
\end{aligned}
$$

- Introducing the Dirichlet-to-Neumann map M for the Laplace problem in $\mathbb{R}^2\backslash\overline{A}_0$ and the ansatz $\psi_1'(t,\mathbf{x}) = \widetilde{\psi}(\mathbf{x})\,e^{\lambda t}$, leads to an eigenvalue problem localized to vortex core A_0

$$
\begin{aligned}
\lambda\widetilde{\psi} &= \Delta_M^{-1}\left[\left(\boldsymbol{\nabla}^{\perp}\psi_0\right)\cdot\boldsymbol{\nabla}\left(\Delta\widetilde{\psi} + b^2\widetilde{\psi}\right)\right] =: \mathcal{L}\widetilde{\psi} && \text{in } A_0,\\
\frac{\partial\Delta\widetilde{\psi}}{\partial r} &= 0, && \text{at } r = 0
\end{aligned}
$$

Δ_M^{-1} is the inverse Laplacian subject to the boundary condition $\partial\widetilde{\psi}/\partial n - M\widetilde{\psi} = 0$

- Introducing the ansatz $\widetilde{\psi}(r,\theta) = \sum_{m=0}^{\infty} f_m(r)\cos(m\theta) + g_m(r)\sin(m\theta)$, the eigenvalue problem becomes for $m = 0, 1, \dots$

$$
\begin{aligned}
\lambda \mathcal{B}_m f_m =& \frac{1}{2} P(r) \frac{d}{dr}\left(\mathcal{B}_{m-1} f_{m-1} + b^2 f_{m-1} + \mathcal{B}_{m+1} f_{m+1} + b^2 f_{m+1}\right) \\
& \frac{1}{2} Q(r) \frac{m}{r}\left(\mathcal{B}_{m-1} f_{m-1} + b^2 f_{m-1} - \mathcal{B}_{m+1} f_{m+1} - b^2 f_{m+1}\right), && r \in (0,1), \\
f_m & \quad \text{bounded} && \text{at } r = 0, \\
\frac{d}{dr} f_m =& -m f_m && \text{at } r = 1, \\
\frac{d}{dr} \mathcal{B}_m f_m =& 0 && \text{at } r = 0
\end{aligned}
$$

where $\mathcal{B}_m f := \frac{d^2}{dr^2} f + \frac{1}{r}\frac{d}{dr} f - \frac{m^2}{r} f$, $\quad P(r) = \frac{2UJ_1(br)}{bJ_0(b)r}$, $\quad Q(r) = -\frac{2U\left[J_0(br) - \frac{J_1(br)}{br}\right]}{J_0(b)}$
(the same system satisfied by the even and odd component, $f_m(r)$ and $g_m(r)$)

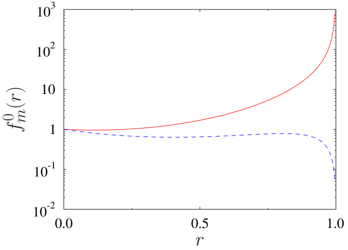
- Consider asymptotic expansions for $m \to \infty$

$$
\lambda = \lambda^0 + \frac{1}{m}\lambda^1 + \mathcal{O}\left(\frac{1}{m^2}\right), \qquad f_m(r) = f_m^0(r) + \frac{1}{m} f_m^1(r) + \mathcal{O}\left(\frac{1}{m^2}\right)
$$

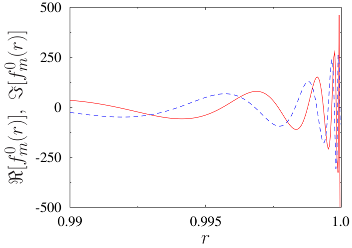
- $\mathcal{O}(m^3)$: $f^0_{m-1}(r) = f^0_m(r) = f^0_{m+1}(r)$

- $\mathcal{O}(m^2)$: $f^0_m(r) = \exp\left[i\int_0^r I_i(r')\,dr'\right]\exp\left[\int_0^r I_r(r')\,dr'\right]$, where

$$I_r(r) := \frac{\Re(\lambda^0)bJ_0(b)r^2 - 4UbJ_0(br)r + 8UJ_1(br)}{2UJ_1(br)r}, \quad I_i(r) := \frac{\Im(\lambda^0)bJ_0(b)r}{2UJ_1(br)}$$



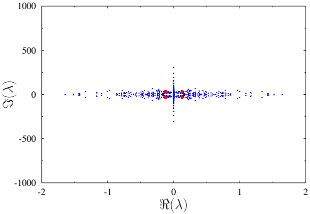
(a) $\lambda^0 = 2$, $\lambda^0 = 6$



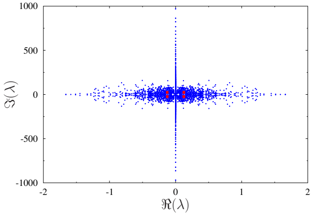
(b) $\lambda^0 = 3 + 10i$

Radial dependence of the eigenvectors $f^0_m(r)$ for different approximate eigenvalues λ^0

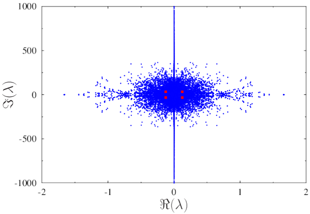
- The approximate eigenfunctions $\widetilde{\psi}(r,\theta)$ are thus dominated by short-wavelength oscillations in r and θ localized near the boundary ∂A_0



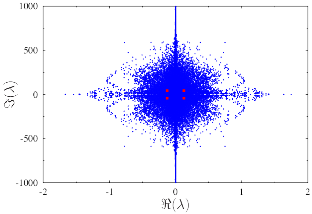
(a) $N = 40$



(b) $N = 80$

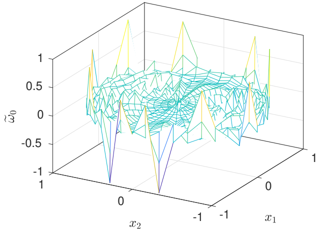


(c) $N = 160$

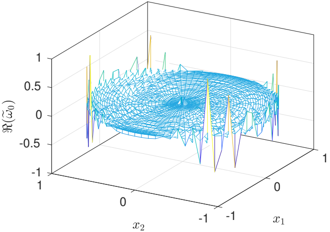


(d) $N = 260$

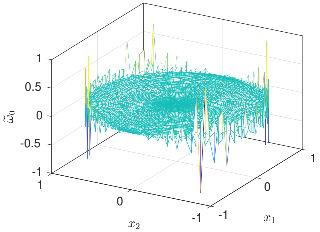
Eigenvalues of the discrete eigenvalue problem with different resolutions N



(a) $N = 40$

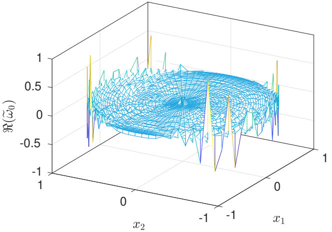


(b) $N = 80$

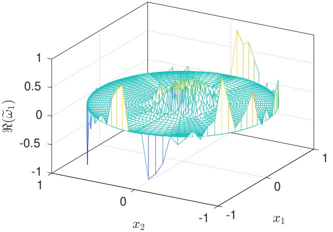


(c) $N = 120$

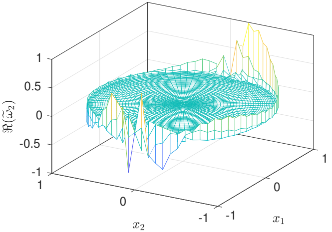
Eigenvectors corresponding to $\lambda_0 = 0.126 \pm i\,25.258$ for different resolutions N



(a) $\lambda_0 = 0.126 \pm i\,25.258$

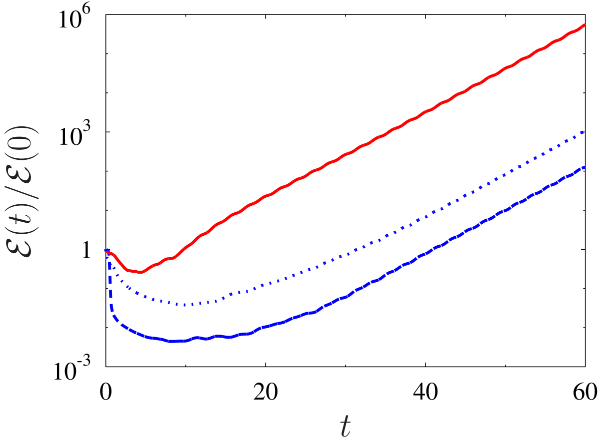


(b) $\lambda_1 = 1.585$



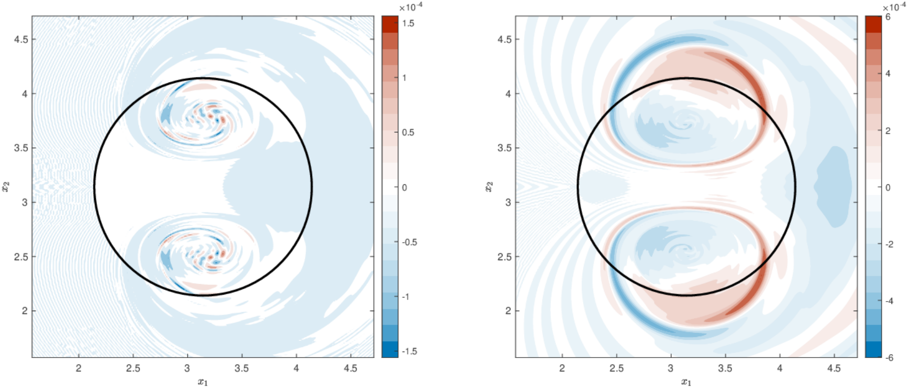
(c) $\lambda_2 = \pm i\,149.873$

Real parts of the eigenvectors corresponding to the indicated eigenvalues



Time evolution of the normalized perturbation enstrophy $\mathcal{E}(t) = \int_\Omega (\omega(t,\mathbf{x}) - \omega_0(\mathbf{x}))^2 \, d\Omega$ in the flow with the initial condition given in terms of (red) the eigenvector $\widetilde{\omega}_0$ and (blue) the eigenvectors $\widetilde{\omega}_1$, $\widetilde{\omega}_2$.

The asymptotic (as $t \to \infty$) growth rate: $d\mathcal{E}(t)/dt \approx \lambda_0 = 0.126$



(a) $t = 4$ (b) $t = 21$

Perturbation vorticity $\overline{\omega}(t, \mathbf{x})$ in the flow corresponding the initial condition involving the eigenvector $\widetilde{\omega}_0$ during (a) the transient regime and (b) the period of exponential growth.

Conclusions

- The Taylor-Green vortex is linearly unstable
  - it has a eigenmode given by a distribution with the Sobolev regularity consistent with Lin (2004)
    - the unstable eigenmode leads to a nonlinear instability
    - computer-assited proof by Cao-Labora et al. (2026)
  - it also exhibits a nonmodal growth mechanism involving a continuous family of uncorrelated functions which saturates the spectral bound
    - this mechanism is consistent with the WKB analysis based on the bicharacteristic equation
- The Lamb-Chaplygin dipole has one linearly unstable eigenmode $\widetilde{\omega}_0$
  - it is associated with the eigenvalue λ_0 embedded in the essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$
  - the eigenmode is again non-smooth, i.e., is dominated by short-wavelength oscillations localized near the boundary ∂A_0
- Open Question: stability of equilibria in 3D