Superfluidity and phase transition in discrete clock symmetry

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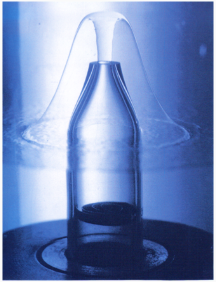
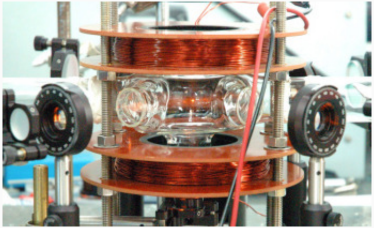
Superfluidity : Inviscid flows in macroscopic scales

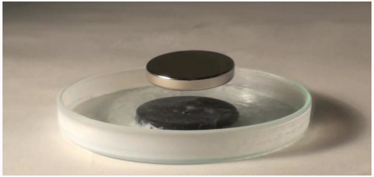
Superfluid He



Atomic BEC

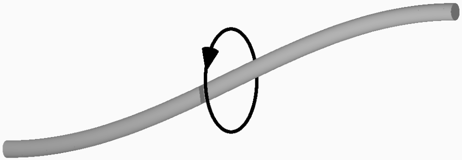


Superconductors



Quantized vortices

Rotational flows are carried by quantized vortices



$$\Gamma = \oint d\boldsymbol{\ell} \cdot \boldsymbol{v} = \frac{2\pi n\hbar}{m} = n\kappa$$

- Quantization of circulation (κ)
- Topological defects
- Very thin cores
- $n = 1$ is dynamically and thermodynamically stabilized

  $\sim$ Å : ^{4}He

  $\sim$ 10 nm : ^{3}He

  $\sim$ 100 nm : Atomic BEC

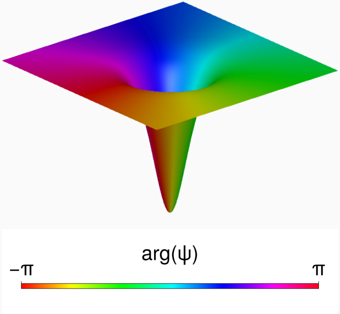
Quantized vortex

Superfluid wave function ψ

Superfluid density : $\rho = |\psi|^2$ Superfluid velocty : $\boldsymbol{v} = \nabla \mathrm{Arg}[\psi]$

Quantized vortex as topological defects

$n = 1$



Quantization of circulation

$$\Gamma = \oint d\boldsymbol{\ell} \cdot \boldsymbol{v} = 2\pi n$$

Discrete symmetry in superfluid

Some superfluid system carry internal discrete symmetry

- Multicomponent BEC with internal Josephson coupling
- Spinor BEC
- Multi-band superconductors
- Superfluid ^{3}He
- Pair superfluid
- charge-4e superconductors

Topological defects induced by discrete symmetry

- Fractionally quantized vortex
- Domain wall

Example: Half-quantized vortex with circulation $n = \frac{1}{2}$

2-component wave function: $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ $\Rightarrow$ Superfluid mass velocity: $\boldsymbol{v} = \dfrac{\rho_1 \boldsymbol{v}_1 + \rho_2 \boldsymbol{v}_2}{\rho_1 + \rho_2}$

Balanced case with $\rho_1 \sim \rho_2 \sim \rho/2$ and $m_1 = m_2 = m$

$$\boldsymbol{v} \sim \frac{\boldsymbol{v}_1 + \boldsymbol{v}_2}{2} = \frac{1}{2}\left(\mathrm{Arg}[\psi_1] + \mathrm{Arg}[\psi_2]\right) = \frac{1}{2}\mathrm{Arg}[\psi_1 \psi_2]$$

$$\Big\downarrow \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} f_1(r) e^{i\theta} \\ f_2(r) \end{pmatrix}$$

$$\Gamma = \oint d\boldsymbol{\ell} \cdot \boldsymbol{v} = \pi$$

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Minimal scalar model for superfluid with discrete symmetry

We suggest two minimal scalar models for discrete symmetry

Model A: $\mathbb{Z}_n$ Goldstone model

$$H = \int d\boldsymbol{x} \, \left\{ a|\nabla\psi|^2 + \frac{b}{n}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\}$$

Model B: Continuous clock model

$$H = \int d\boldsymbol{x} \, \left\{ |\nabla\psi|^2 + \frac{g_1}{2}(|\psi|^2 - \rho)^2 - \frac{g_2}{n}(\psi^n + \psi^{*n}) \right\}$$

- Model A: spontaneous emergence of a discrete degree of freedom from competing gradient terms.
- Model B: explicit introduction of a discrete anisotropy in the order-parameter potential.

Comparing two limits for $n = 2$

$\mathbb{Z}_2$ Goldstone models

$$H = \int dx \left\{ a|\nabla\psi|^2 + \frac{b}{2}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\}$$

$a = 1$, $b = 0$: Standard Gross-Pitaevskii model $\quad \dfrac{G}{H} \simeq U(1)$

$$H = \int dx \left\{ |\nabla\psi|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\}$$

$a = 0$, $b = 1$: π-phase winding model $\quad \dfrac{G}{H} \simeq \dfrac{U(1)}{\mathbb{Z}_2}$

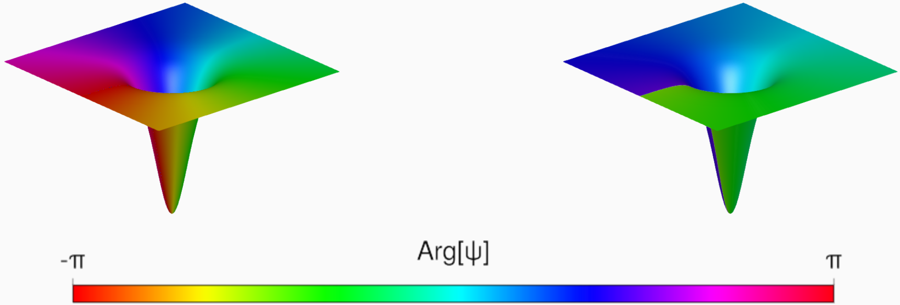
$$H = \int dx \left\{ \frac{1}{2}|\nabla\psi^2|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\}$$

Half-quantized vortex in Model A

Comparing two limits for $n = 2$

$\mathbb{Z}_2$ Goldstone models

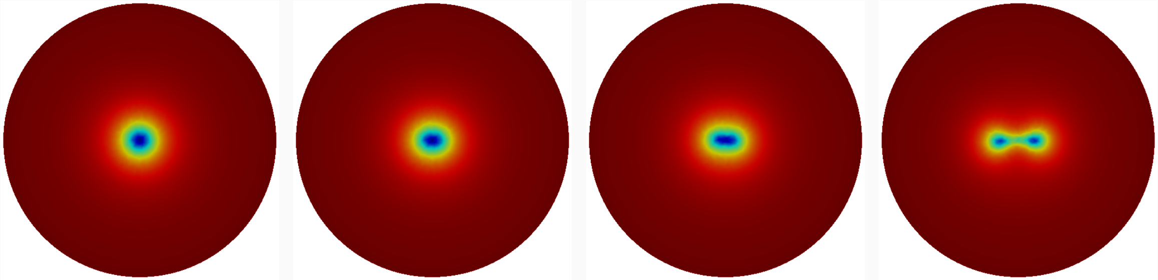
$$H = \int d\boldsymbol{x} \left\{ |\nabla\psi|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\} \Leftrightarrow H = \int d\boldsymbol{x} \left\{ \frac{1}{2}|\nabla\psi^2|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\}$$



Result for the stable solution

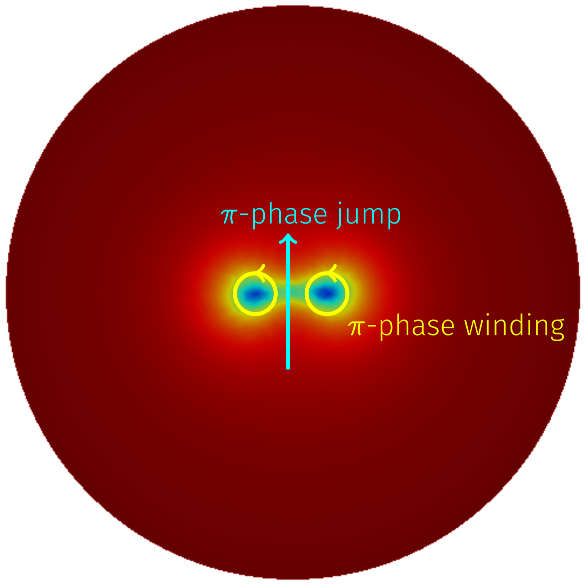
$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + \frac{b}{2}|\nabla\psi^2|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\} \qquad a = \cos\theta \quad b = \sin\theta$$

$\theta = 82^\circ$ $\theta = 84^\circ$ $\theta = 86^\circ$ $\theta = 88^\circ$



Integer vortex splits into two half-quantized vortex connected by a domain wall at $b \gg a \neq 0$

Molecule state with half-quantized vortices

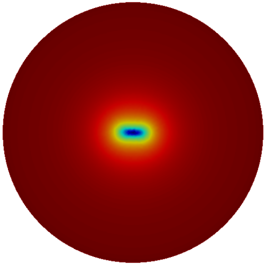


Model A with $n = 3$

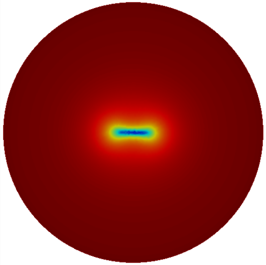
$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + \frac{b}{3}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\}$$

$n = 3$ case : molecule state with one-third vortices

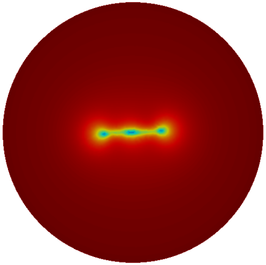
$\theta = 80^\circ$



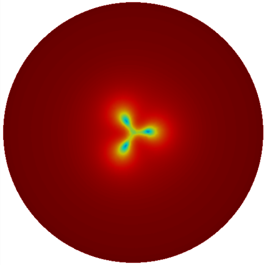
$\theta = 84^\circ$



$\theta = 88^\circ$



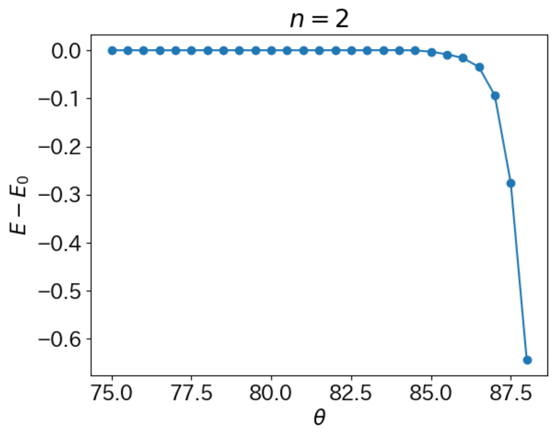
$\theta = 88^\circ$ (metastable)

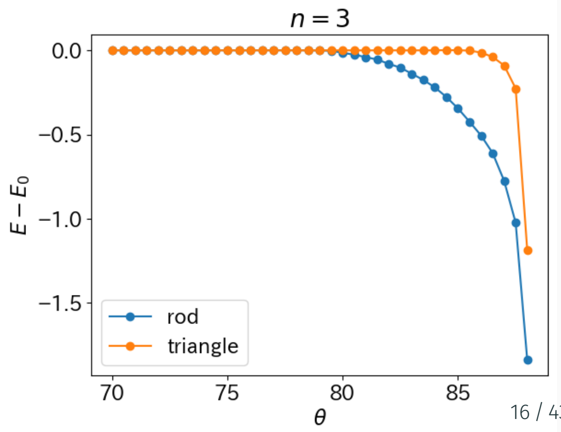


Energy profile

$\mathbb{Z}_n$ modified Goldstone model

$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + b|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - 1)^2 \right\}$$





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Berezinskii-Kosterlitz-Thouless transition

Thermodynamic phase transition in 2-dimension with continuous symmetry

- No thermodynamic singularity (smooth free energy)
- No symmetry breaking
- Singularity for linear response (jump of helicity modulus)
- Transition from free vortices to bounded vortex-anti vortex pairs

Stochastic Gross-Pitaevskii equation

Stochastic GP equation: finite-temperature model in equilibrium

$$
\begin{aligned}
(i-\gamma)\Delta\psi &= \frac{\delta H}{\delta\psi^*}\Delta t + \sqrt{\gamma T(\Delta x)(\Delta t)}(\Delta W_1 + i\Delta W_2) \\
&= \left[-a\nabla^2\psi + b\{(\nabla\psi^2)\nabla\psi^* - (\nabla^2\psi^2)\psi^*\} + g(|\psi|^2-\rho)\psi\right]\Delta t \\
&+ \sqrt{\gamma T(\Delta x)(\Delta t)}(\Delta W_1 + i\Delta W_2)
\end{aligned}
$$

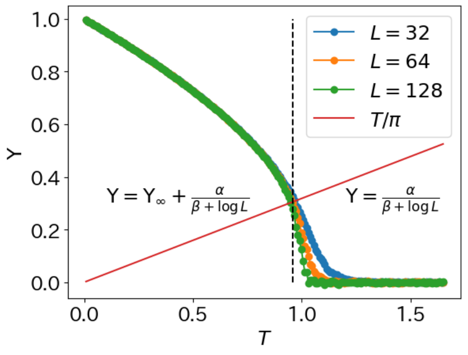
$$
\langle\Delta W_i\rangle = 0 \quad \langle(\Delta W_i)(\Delta W_j)\rangle = \delta_{i,j} \;\Rightarrow\; \langle O\rangle = \frac{1}{T}\lim_{T\to\infty}\int_0^T dt\, O = \int D\psi D\psi^*\, O e^{-H/T}
$$

Helicity modulus

Helicity modulus

Twisted boundary condition : $\psi(x+L, y) = e^{i\delta L}\psi(x, y)$

Helicity modulus obtained from free-energy difference : $\Upsilon = \dfrac{F(\delta) - F(0)}{\delta^2}$

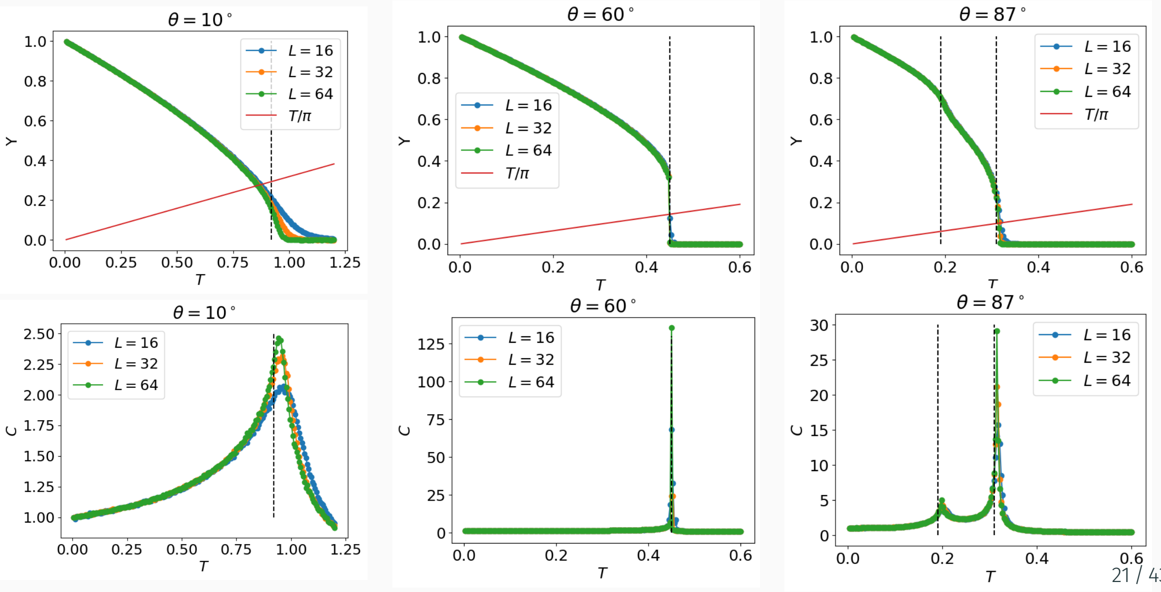


Kosterlitz-Thouless transition of standard GP model

Universal relation of Helicity modulus

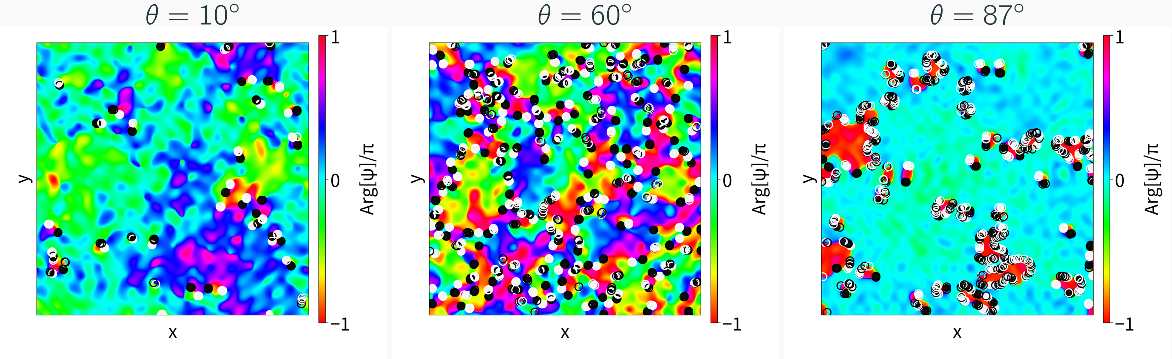
$$\frac{\Delta\Upsilon}{T_{\mathrm{KT}}} = \frac{1}{\pi}$$

Helicity modulus and specific heat

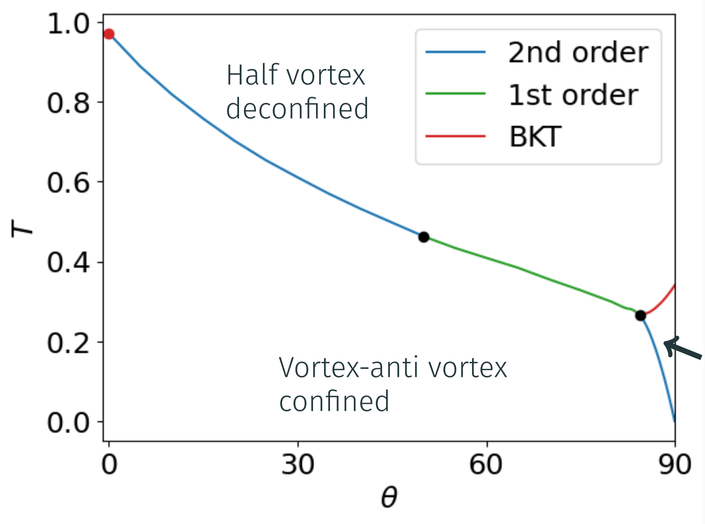


Phase and vortex snapshot

$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + \frac{b}{2}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\} \qquad a = \cos\theta \quad b = \sin\theta$$



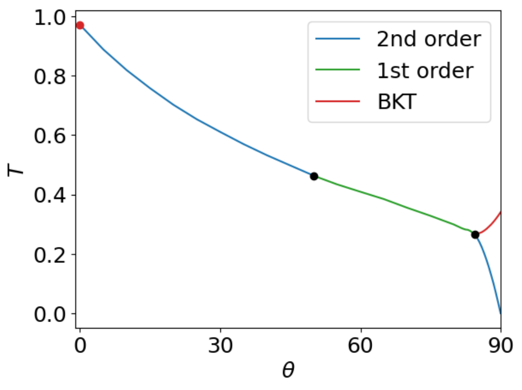
θ-T Phase diagram

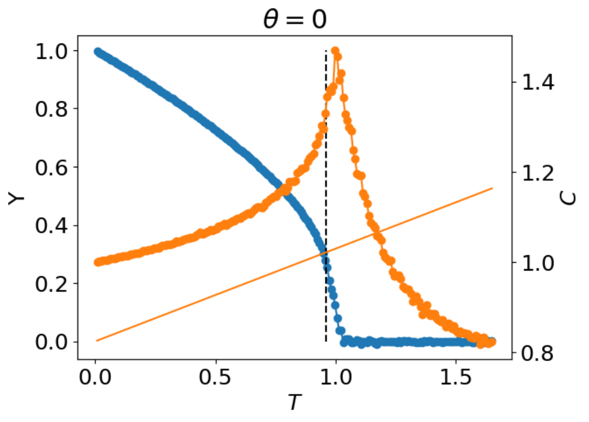


Half vortex
confined

$\theta = 0$: Standard BKT transition

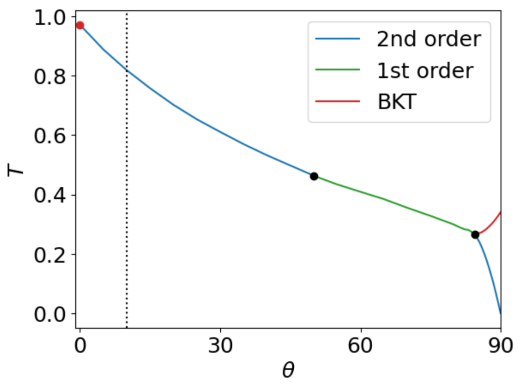
$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + \frac{b}{2}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\} \qquad a = \cos\theta \quad b = \sin\theta$$

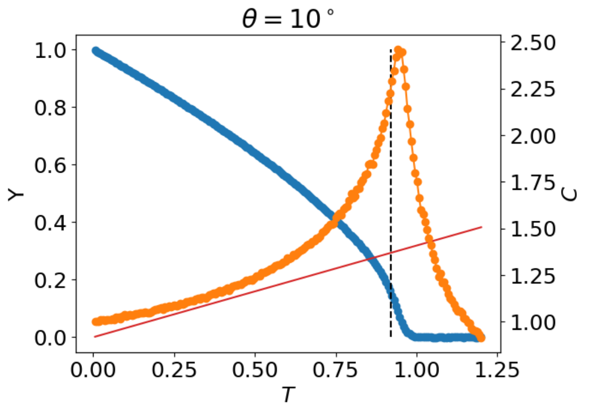




Small θ : 2nd ordered transition

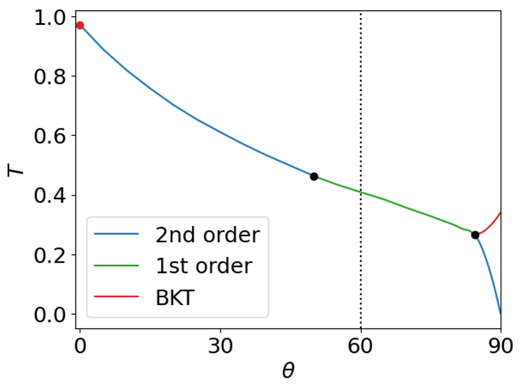
$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + \frac{b}{2}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\} \qquad a = \cos\theta \quad b = \sin\theta$$

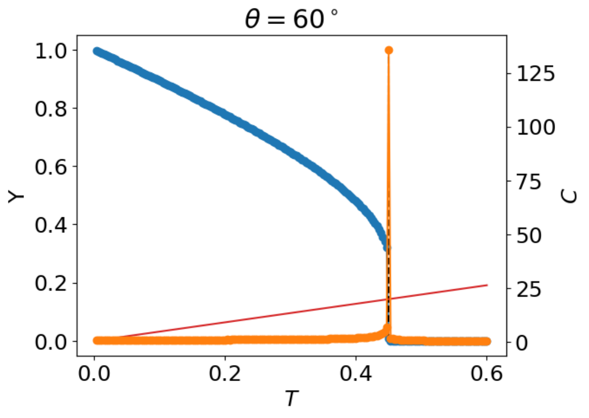




Intermediate θ : 1st ordered transition

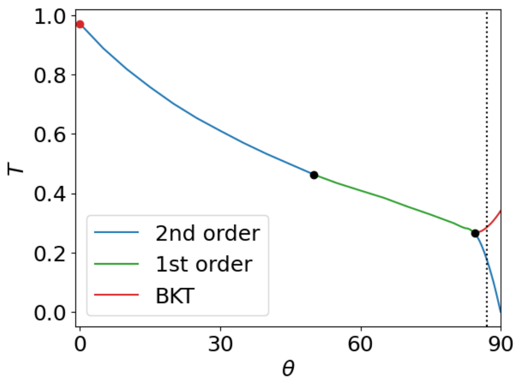
$$H = \int dx \left\{ a|\nabla\psi|^2 + \frac{b}{2}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\} \qquad a = \cos\theta \quad b = \sin\theta$$

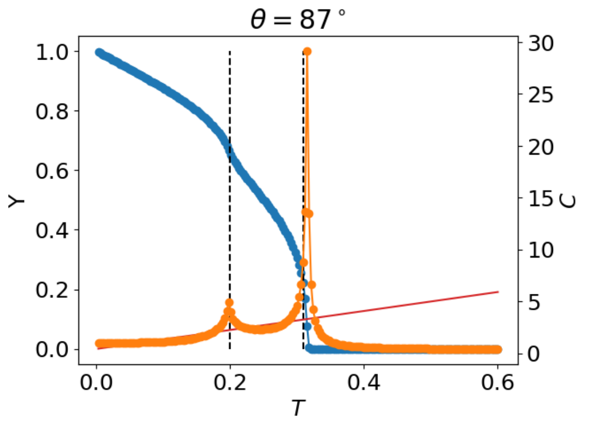




Large θ : BKT (high T) and 2nd ordered (low T) transitions

$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + \frac{b}{2}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\} \qquad a = \cos\theta \quad b = \sin\theta$$





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Model B: continuous clock model

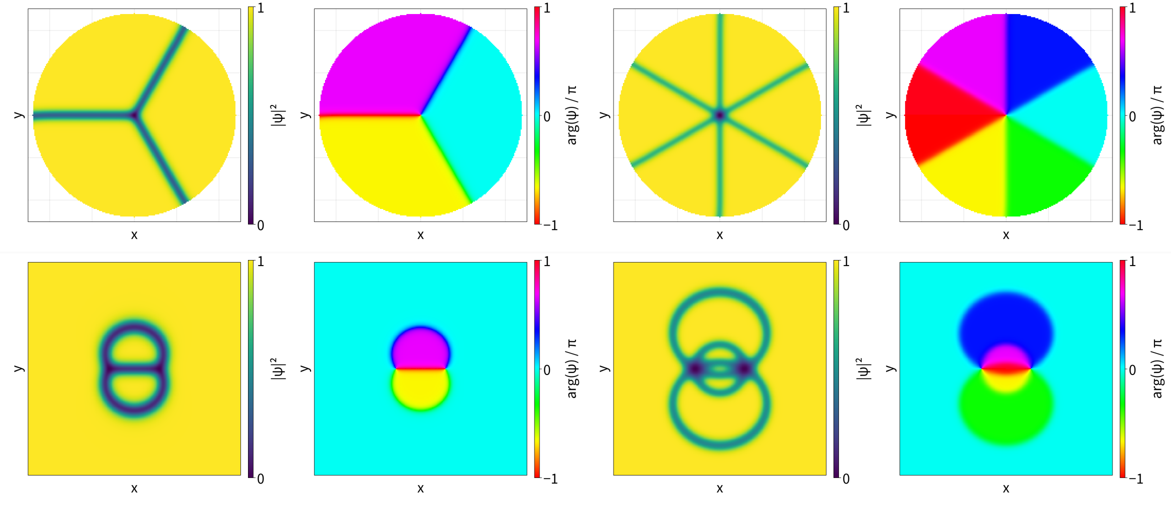
Model B

$$H = \int d\boldsymbol{x} \left\{ |\nabla\psi|^2 + \frac{g_1}{2}(|\psi|^2 - \rho)^2 - \frac{g_2}{n}(\psi^n + \psi^{*n}) \right\}$$

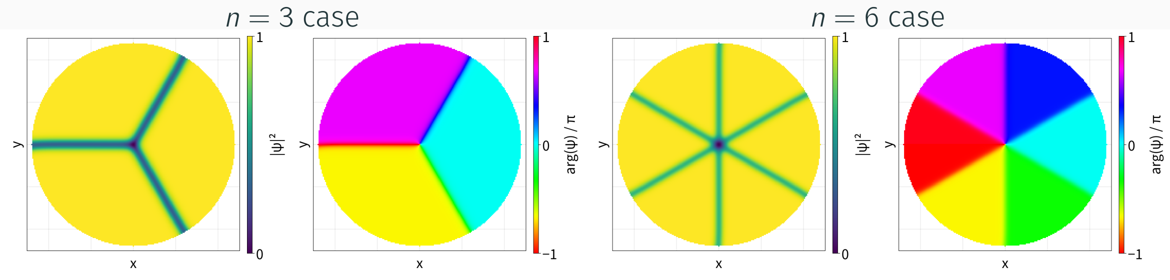
$\arg(\psi) = \dfrac{2\pi}{n}$ is favored $\qquad \dfrac{G}{H} \simeq \mathbb{Z}_n$

Vortex and vortex pair solution in model B

$n = 3$ case $\qquad$ $n = 6$ case



Vortex and vortex pair solution in model B



n domain walls are attached to vortex

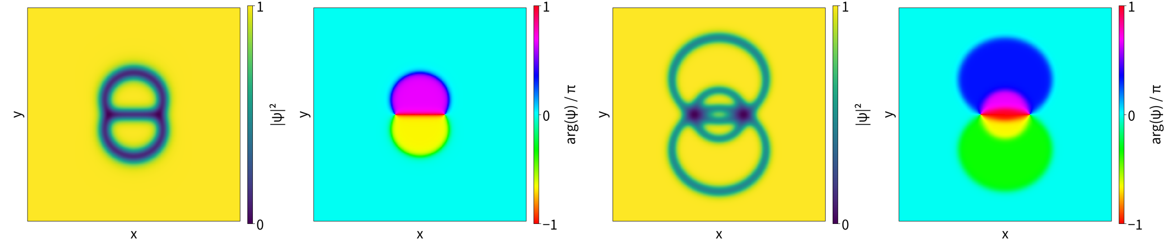
Vortex cannot exist in bulk due to domain wall

Vortex and vortex pair solution in model B

n domain walls are attached to vortex

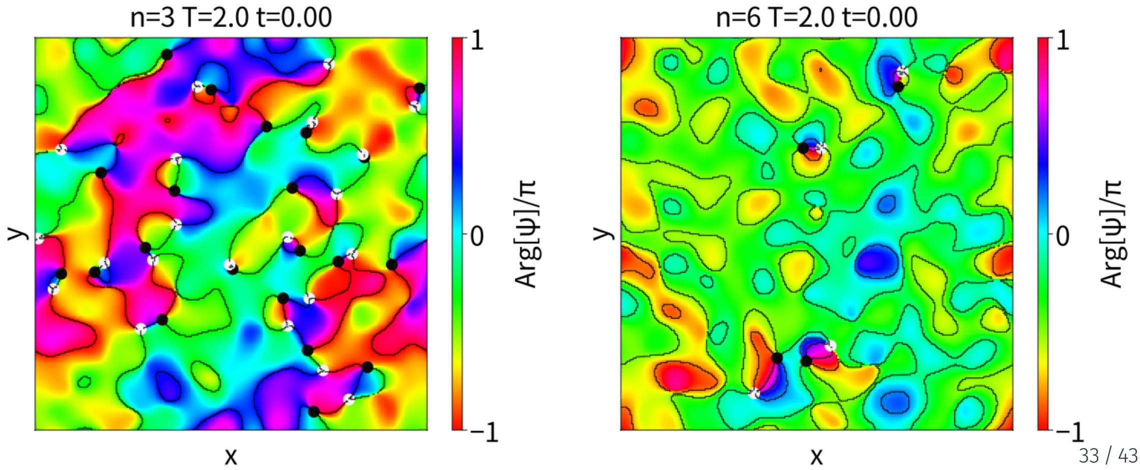
A vortex pair can exist in bulk

$n = 3$ case $n = 6$ case



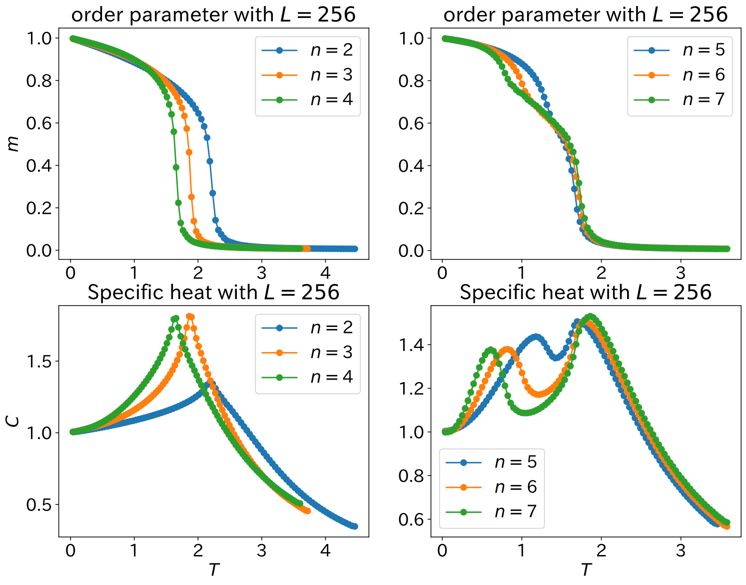
Stochastic Gross-Pitaevskii equation in equilibrium

$$(i-\gamma)\Delta\psi = \left\{-\nabla^2\psi + g_1(|\psi|^2-\rho)\psi - g_2\psi^{*n-1}\right\}\Delta t + \sqrt{\gamma T(\Delta x)(\Delta t)}(\Delta W_1 + i\Delta W_2)$$



Large-g_1 case: $H = \int dx \left\{ |\nabla\psi|^2 + \frac{g_1}{2}(|\psi|^2 - \rho)^2 - \frac{g_2}{n}(\psi^n + \psi^{*n}) \right\}$

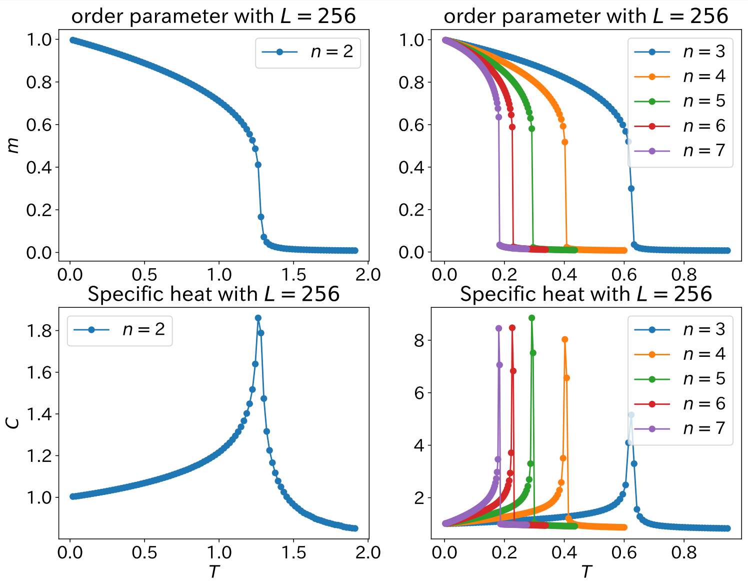
Single transition (2nd order)



Double BKT transition

Small-g_1 case: $H = \int dx \left\{ |\nabla\psi|^2 + \frac{g_1}{2}(|\psi|^2 - \rho)^2 - \frac{g_2}{n}(\psi^n + \psi^{*n}) \right\}$

Single transition (2nd order)

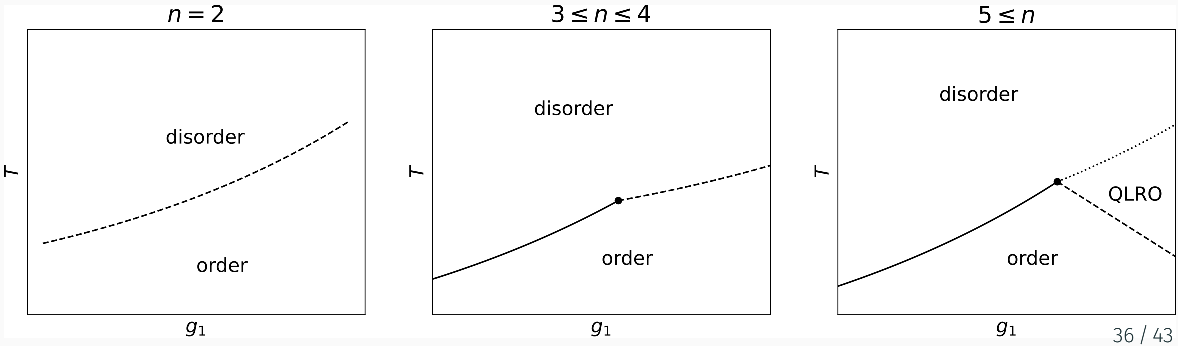


Single transition (1st order)

Details for phase transition

$$H = \int d\boldsymbol{x} \left\{ |\nabla\psi|^2 + \frac{g_1}{2}(|\psi|^2 - \rho)^2 - \frac{g_2}{n}(\psi^n + \psi^{*n}) \right\}$$

	$n = 2$	$n = 3, 4$	$n \geq 5$
Large g_1	Single 2nd order		Double BKT
Small g_1	Single 2nd order	Single 1st order	



Discussion: two routes to discrete-symmetry physics

We studied two minimal scalar models to isolate the role of discrete symmetry in superfluid phase transitions.

Model A: $\mathbb{Z}_n$ Goldstone model

$$H = \int d\boldsymbol{x} \left\{ a|\nabla\psi|^2 + \frac{b}{n}|\nabla\psi^n|^2 + \frac{g}{2}(|\psi|^2 - \rho)^2 \right\}$$

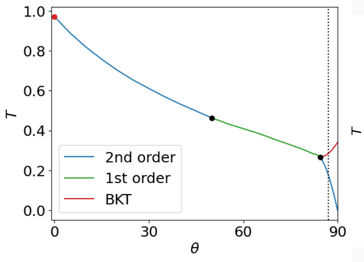
Model B: Continuous clock model

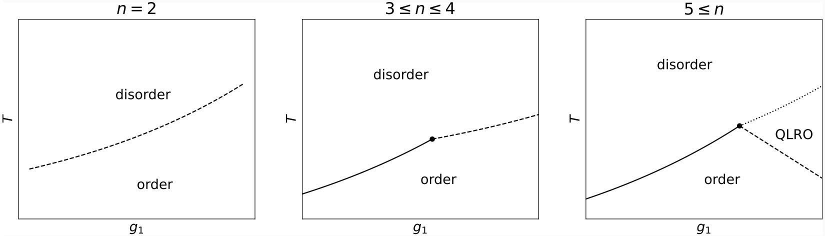
$$H = \int d\boldsymbol{x} \left\{ |\nabla\psi|^2 + \frac{g_1}{2}(|\psi|^2 - \rho)^2 - \frac{g_2}{n}(\psi^n + \psi^{*n}) \right\}$$

- Model A: competing gradient terms generate fractional vortices and an emergent pair-order-like intermediate phase.
- Model B: explicit $\mathbb{Z}_n$ anisotropy confines vortices by domain walls and produces n-dependent phase-transition scenarios.

Discussion and summary: Transition scenarios found in the two models

- First-order phase transition (discrete)
- Second-order phase transition (continuous)
- BKT transition (quasi-long-range order)





Discussion and summary: Possible connections and outlook

- Multicomponent BEC with internal Josephson coupling
- Spinor BEC
- Superfluid ^{3}He
- Charge-$4e$ superconductors
- Pair superfluid
- Unconventional superconductors with strong crystal anisotropy

A future direction is to clarify how these scalar-model scenarios are realized in realistic multicomponent systems.

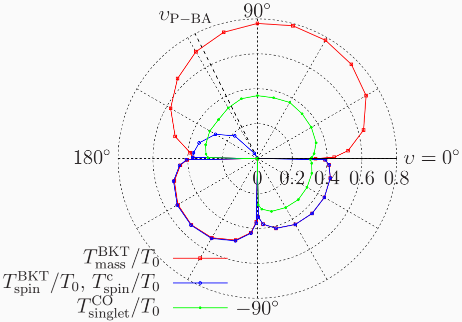
Michikazu Kobayashi and Muneto Nitta
Phys. Rev. D 101, 085003-1-6 (2020) / arXiv:1912.09456

Michikazu Kobayashi, Gergely Fejős, Chandrasekhar Chatterjee, and Muneto Nitta
Phys. Rev. Research. 2, 013081-1-14 (2020) / arXiv:1908.11087

Buckup: Spin-1 spinor BEC
MK, J. Phys. Soc. Jpn. 88, 094001 (2019).

2-dimensional spin-1 spinor BEC: $\psi = \begin{pmatrix} \psi_1 & \psi_0 & \psi_{-1} \end{pmatrix}$

$$H = \int dx \left\{ \sum_{s=-1}^{1} (|\nabla \psi_s|^2 + q s^2 |\psi_s|^2) + \frac{g_0}{2}(\psi^\dagger \psi)^2 + \frac{g_1}{2}(\psi^\dagger \boldsymbol{S} \psi)^2 \right\}$$



Phase diagram for $\tan \upsilon = \dfrac{g_2}{g_1 \psi^\dagger \psi}$

$0 < \upsilon < \upsilon_{\mathrm{P-BA}}$ $\qquad \dfrac{G}{H} \simeq U(1)$

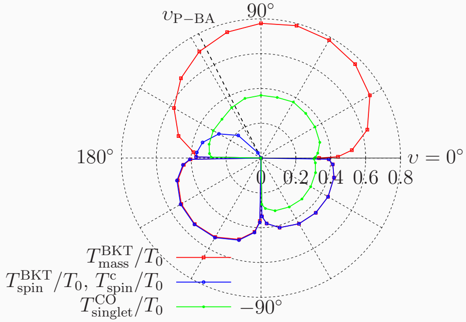
Polar state → Standard BKT transition

Mass integer vortex

Buckup: Spin-1 spinor BEC MK, J. Phys. Soc. Jpn. 88, 094001 (2019).

2-dimensional spin-1 spinor BEC: $\psi = \begin{pmatrix}\psi_1 & \psi_0 & \psi_{-1}\end{pmatrix}$

$$H = \int d\boldsymbol{x} \left\{ \sum_{s=-1}^{1} (|\nabla \psi_s|^2 + q s^2 |\psi_s|^2) + \frac{g_0}{2}(\psi^\dagger \psi)^2 + \frac{g_1}{2}(\psi^\dagger \boldsymbol{S} \psi)^2 \right\}$$



Phase diagram for $\tan\upsilon = \dfrac{g_2}{g_1 \psi^\dagger \psi}$

$\upsilon < \upsilon_{\mathrm{P-BA}} < \pi$ $\dfrac{G}{H} \simeq U(1) \times U(1)$

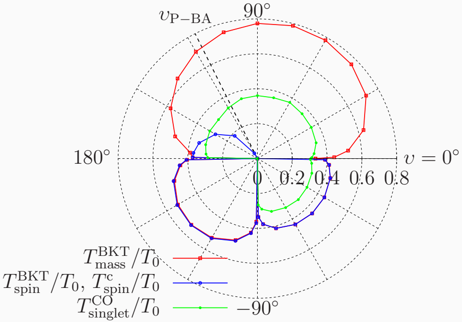
Broken-axisymmetric state

$\rightarrow$ Mass and spin BKT transitions

Mass and spin integer vortices

2-dimensional spin-1 spinor BEC: $\psi = \begin{pmatrix} \psi_1 & \psi_0 & \psi_{-1} \end{pmatrix}$

$$H = \int dx \left\{ \sum_{s=-1}^{1} (|\nabla \psi_s|^2 + qs^2|\psi_s|^2) + \frac{g_0}{2}(\psi^\dagger \psi)^2 + \frac{g_1}{2}(\psi^\dagger \boldsymbol{S} \psi)^2 \right\}$$



Phase diagram for $\tan \upsilon = \dfrac{g_2}{g_1 \psi^\dagger \psi}$

$$-\pi < \upsilon < -\pi/2 \qquad \frac{G}{H} \simeq O(2)$$

Ferromagnetic state

→ Mass BKT and spin 2nd ordered transitions

Mass vortex and spin domain

Buckup: Spin-1 spinor BEC

MK, J. Phys. Soc. Jpn. 88, 094001 (2019).

2-dimensional spin-1 spinor BEC: $\psi = \begin{pmatrix} \psi_1 & \psi_0 & \psi_{-1} \end{pmatrix}$

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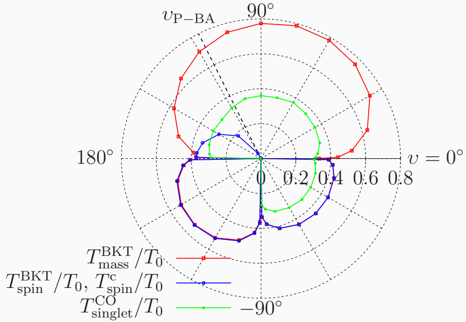

Phase diagram for $\tan \upsilon = \dfrac{g_2}{g_1 \psi^\dagger \psi}$

$-\pi/2 < \upsilon < 0 \qquad \dfrac{G}{H} \simeq \dfrac{U(1) \times U(1)}{\mathbb{Z}_2}$

Anti-ferromagnetic state

$\rightarrow$ Mass-spin coupled BKT transition

Half-quantized vortex