Quantum turbulence in He-II: HVBK cascade, mutual friction and Reynolds-number effects

From component budgets to mixture dynamics

L. Danaila
Z. Zhang, I. Danaila, E. Lévêque, J.I. Polanco, P.-E. Roche

Laboratoire M2C, University of Rouen Normandy
LMFA, École Centrale de Lyon, CNRS
Laboratoire de Mathématiques Raphaël Salem

AIMS Conference, Athens, 6-10 July 2026

Core update: the PNAS 2025 paper resolves the apparent temperature dependence of intermittency by introducing the correct two-fluid mixture scales.

HVBK essentials: two coupled turbulent fluids

He-II below Tλ

Normal fluid: viscous, carries entropy.

Superfluid: inviscid at large scales, rotational motion through quantized vortices.

The two fields are coupled by mutual friction.

Central difficulty

Two velocities, two viscosities, one observed turbulent cascade.

The friction term is both a local energy-exchange mechanism and a possible dissipation mechanism.

$$\partial_t \mathbf{u}_{n,s} + \mathbf{u}_{n,s} \cdot \nabla \mathbf{u}_{n,s} = -\nabla p_{n,s} \pm \mathbf{F}_M / \rho_{n,s} + \nu_{n,s} \Delta \mathbf{u}_{n,s} + \mathbf{f}_{n,s}$$

For a classical fluid, the Kármán-Howarth relation connects the third-order structure function to energy dissipation. For He-II, one must decide whether to write this relation for each component or for the mass-weighted mixture.

Why the question remained ambiguous

What earlier studies agreed on

A -5/3 spectrum at large scales.

Strong locking of normal and superfluid velocities.

Intermittency is present.

What remained incompatible

Opposite trends for small scales 'Kolmogorov values, when temperature decreases.

Reported enhancement or depletion of intermittency.

Different Reynolds-number definitions across components.

Clarifying idea:
separate the component-level energy exchange from the mixture-level energy cascade.

This is the conceptual bridge from the JFM 2023 paper to the PNAS 2025 paper.

Talk question: What does mutual friction change - and what does it not change - in the scale-by-scale cascade?

Mutual friction is essential for locking the two components and redistributing energy, but the inertial-range cascade of the He-II mixture remains classical-like when expressed with effective two-fluid scales.

The unresolved question:

Is quantum-turbulence intermittency controlled by temperature, or by Reynolds number?

1) Use the 2023 component budgets to show where mutual friction acts.

2) Then use the 2025 mixture budget to show what survives at the scale-by-scale cascade level.

1) Zhang, Danaila, Lévêque & Danaila, J. Fluid Mech. 962, A22 (2023): component-level higher-order budgets and frictional exchanges.

2) Polanco, Roche, Danaila & Lévêque, PNAS 122, e2426598122 (2025): mixture energy budget, effective microscales and intermittency reinterpretation.

HVBK two-fluid model

Helium II two-fluid model (Tisza & Landau 1941)

- Normal fluid: ρ^n, v_n, ν^n, carries entropy
- Superfluid: ρ^s, v_s, free of entropy
- Below T_λ mixture of two fluid $\rho = \rho^n + \rho^s$, for $T > T_\lambda$, $\rho^s/\rho = 0$; for $T = 0K$, $\rho^n/\rho = 0$

The mutual friction force

$$\boldsymbol{F}_M = -B\frac{\rho^s\rho^n}{\rho}\frac{\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge (\boldsymbol{u}_s - \boldsymbol{u}_n))}{|\boldsymbol{\omega}|} - B'\frac{\rho^s\rho^n}{\rho}\boldsymbol{\omega} \wedge (\boldsymbol{u}_s - \boldsymbol{u}_n) \tag{1}$$

B and $B^{'}$: two temperature-dependent parameters, $\omega = \nabla \times \boldsymbol{u}_s$,
$\boldsymbol{u}_n$: the normal fluid velocity, $\boldsymbol{u}_s$: the superfluid velocity.

The governing equations

$$\nabla.\boldsymbol{u}_n = 0,\ \nabla.\boldsymbol{u}_s = 0 \tag{2}$$

$$\frac{\partial \boldsymbol{u}_n}{\partial t} + \nabla.(\boldsymbol{u}_n \otimes \boldsymbol{u}_n) = -\nabla p_n + \frac{1}{\rho^n}\boldsymbol{F}_M + \nu^n \Delta \boldsymbol{u}_n + \boldsymbol{f}_{ext} \tag{3}$$

$$\frac{\partial \boldsymbol{u}_s}{\partial t} + \nabla.(\boldsymbol{u}_s \otimes \boldsymbol{u}_s) = -\nabla p_s - \frac{1}{\rho^s}\boldsymbol{F}_M + \nu^s \Delta \boldsymbol{u}_s + \boldsymbol{f}_{ext} \tag{4}$$

Objectives of the present research

Mutual friction force

- Statistics of the mutual friction force
- Energy exchange mechanism between each fluid component. Disentanglement of the inner energy exchange

Scaling laws.. if any, and related issues

- Transport equations for the second-order structure functions
- The third-order structure function transfer equation $\rightarrow$ flatness factor

Small-scale intermittency

- Statistics of velocity increments at small scales

The numerical simulations

The 3dt-HVBK solver

A solver of 3D box periodic boundary condition HIT using spectral method (p3dfft, pencil 2D decomposition), large eddy forcing, standard de-aliasing, two-fluid HVBK model.

The truncated HVBK two-fluid model

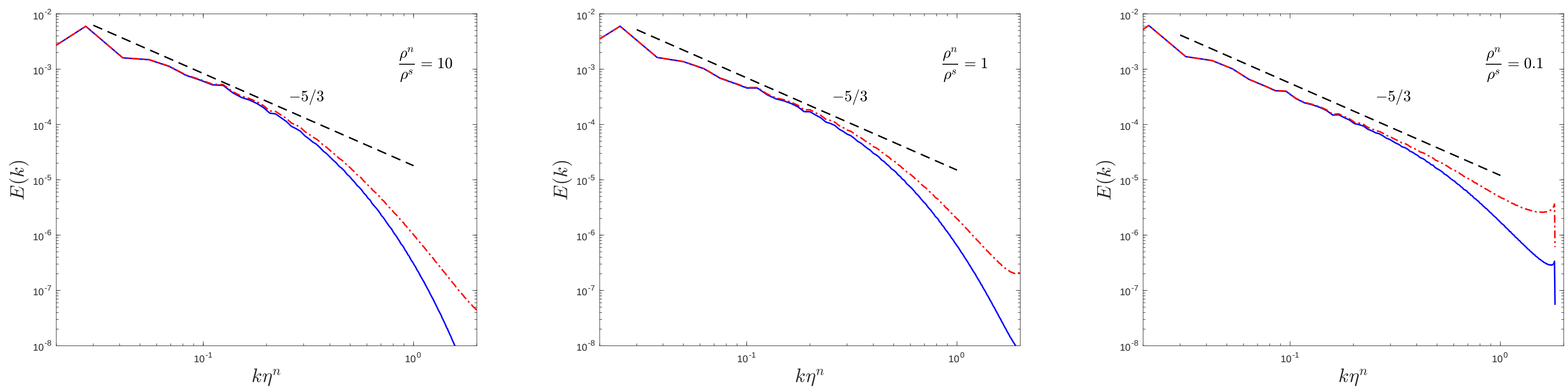
- $\nu^n/\nu^s = 10$ with $\nu^n = cst$ (independent of temperature)
- Simplified mutual friction expression, with $B = 1.5$ invariant to T

$$\boldsymbol{F}_M = \frac{B}{2}\frac{\rho^s\rho^n}{\rho}|\omega|.(\boldsymbol{u}_s - \boldsymbol{u}_n)$$

- Unique variable $\rho^n/\rho^s = 10,\ 1,\ 0.1$, for high, intermediate and low temperature.
- External forcing f^n_{ext} and f^s_{ext} with constant energy injection rate ε^*.

The energy spectrum

- -5/3 in the inertial range
- When temperature decreases: the inertial range extends for both the normal fluid and the superfluid.
- When temperature decreases: the Kolmogorov scales η decrease monotonously.



(a) (b) (c)

Figure: The kinetic energy spectrum for different temperature (a) $\rho^n/\rho^s = 10$, (b) $\rho^n/\rho^s = 1$, (c) $\rho^n/\rho^s = 0.1$. (–) is the normal fluid, ($-\cdot$) is the superfluid, (- -) marks the $-5/3$ slope. Wavenumbers are normalised with the normal fluid Kolmogorov scales, $1/\eta^n$.

The mutual friction force

The energy exchange through the mutual friction

- Knowing that the high enstrophy area are the same, the strong energy exchange only occurs at the strong enstrophy area

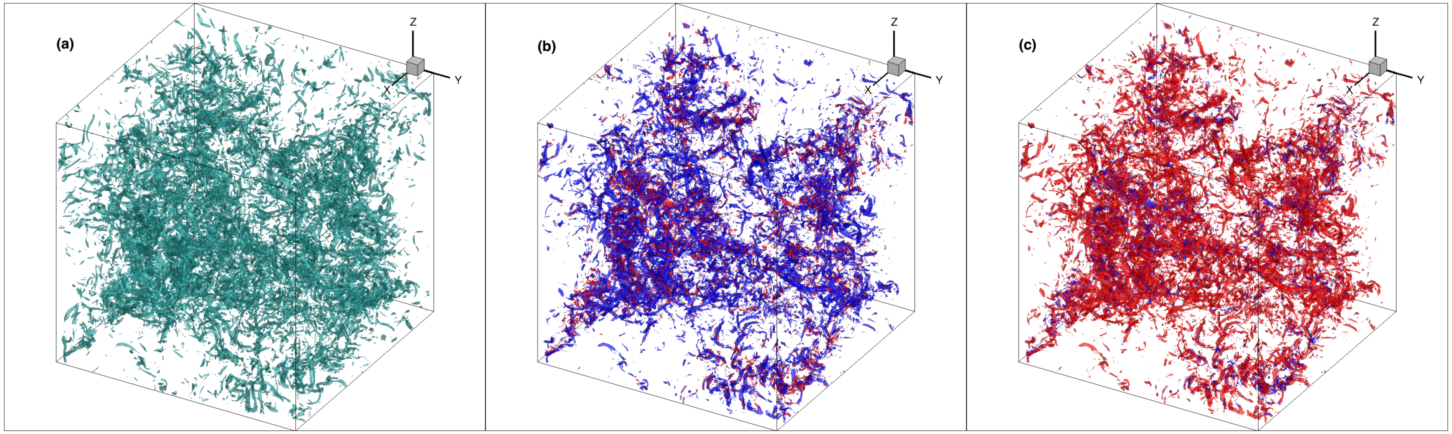
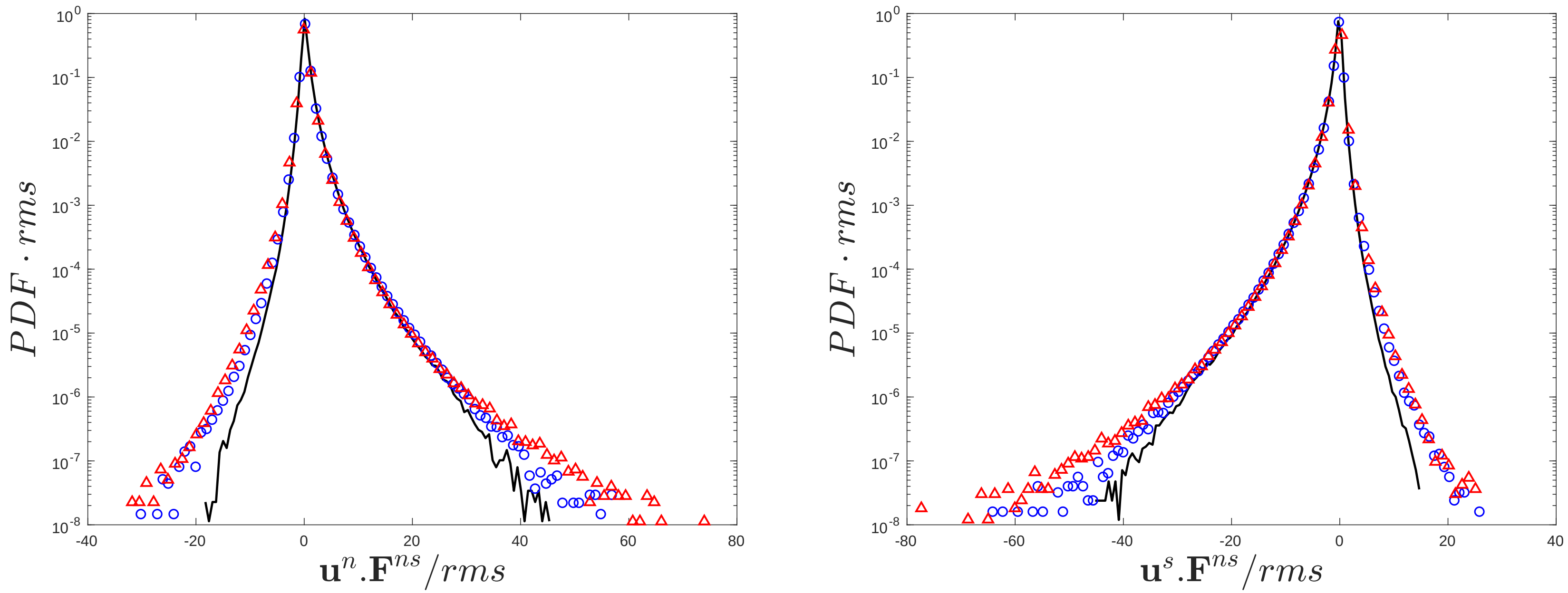


Figure: (a) The snapshot of iso-contour $|\omega_s| = 7\sqrt{\langle|\omega_s|^2\rangle}$, (b) the snapshot of the iso-contour of $u_n.F_{ns}$. (blue) energy gain $5\sqrt{\langle(u_n.F_{ns})^2\rangle}$, (red) energy loss $-3\sqrt{\langle(u_n.F_{ns})^2\rangle}$, (c) the snapshot of the iso-contour of $u_s.F_{ns}$. (blue) energy gain $5\sqrt{\langle(u_s.F_{ns})^2\rangle}$, (red) energy loss $-3\sqrt{\langle(u_s.F_{ns})^2\rangle}$,

The mutual friction force

The energy exchange through the mutual friction

- Skewed PDF form)
- Stretched tails of the PDF (a source term looks like the dissipation rate)



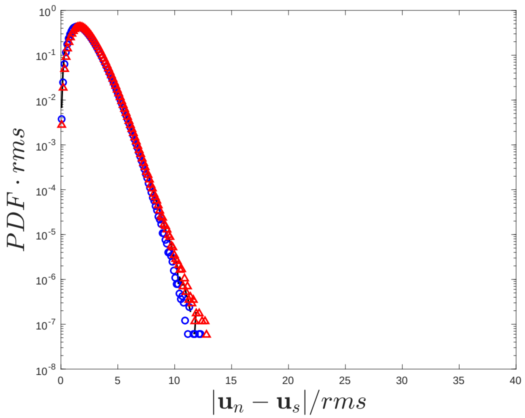
(a) The ϕ^n in the normal fluid

(b) The ϕ^s in the superfluid

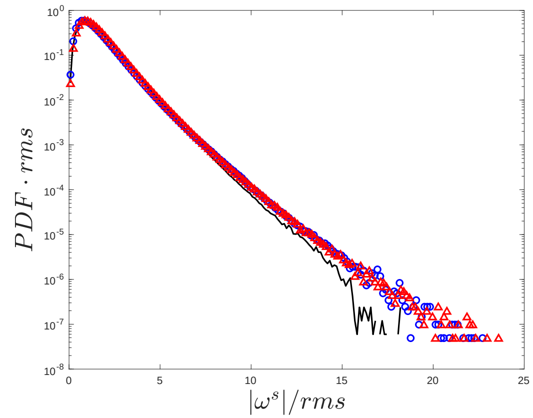
Figure: The PDFs of the energy exchange flux normalised by their RMS values for different temperatures (–) $\rho^n/\rho^s = 10$, (○) $\rho^n/\rho^s = 1$, (△) $\rho^n/\rho^s = 0.1$.

The energy exchange through the mutual friction

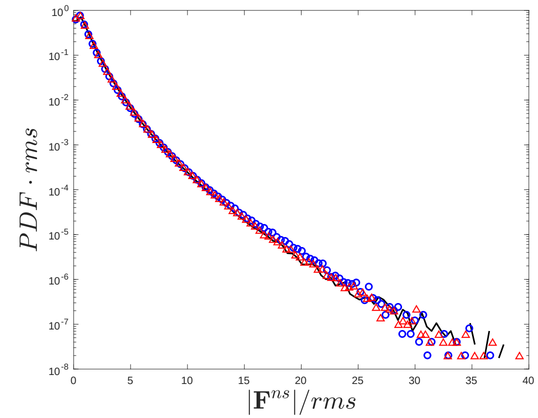
- Non-Gaussian PDF of the mutual friction
- Stretched tails mainly from the enstrophy
- Skewness from the relative velocity



(a) The magnitude of relative velocity



(b) The enstrophy of the superfluid



(c) The magnitude of mutual friction

Figure: The PDFs of the involved variables in the mutual friction force normalised by their RMS values, for different temperature ($-$) $\rho^n/\rho^s = 10$, ($\circ$) $\rho^n/\rho^s = 1$, ($\triangle$) $\rho^n/\rho^s = 0.1$.

We follow the classical approach in HIT for the second–order structure function transport equations, and integration from 0 to r leads to the 4/3 law, viz.

$$-\overline{\delta u_{\parallel}^{n}(\delta u_i^{n})^2} - \frac{1}{r^2}\int_0^r s^2 \mathcal{L}^n ds + 2\nu^n \frac{d}{dr}\overline{(\delta u_i^n)^2} = \frac{4}{3}\overline{\epsilon}^n r, \tag{5}$$

$$-\overline{\delta u_{\parallel}^{s}(\delta u_i^{s})^2} - \frac{1}{r^2}\int_0^r s^2 \mathcal{L}^s ds + 2\nu^s \frac{d}{dr}\overline{(\delta u_i^s)^2} = \frac{4}{3}\overline{\epsilon}^s r, \tag{6}$$

with

$$\mathcal{L}^n \equiv -2\frac{\rho^s}{\rho}\overline{(\delta u_i^n)(\delta F_i^{ns})} - 2\overline{(\delta u_i^n)(\delta f_i^n)}, \tag{7}$$

$$\mathcal{L}^s \equiv 2\frac{\rho^n}{\rho}\overline{(\delta u_i^s)(\delta F_i^{ns})} - 2\overline{(\delta u_i^s)(\delta f_i^s)}, \tag{8}$$

where $u_{\parallel}$ is the velocity component parallel to vector $\vec{r}$, $r = |\vec{r}|$, $\delta f = f(\vec{x}+\vec{r}) - f(\vec{x})$ and for isotropic turbulence δf depends only on r.

The scale-by-scale energy budget

$$\underbrace{-\overline{\delta u_{\|}(\delta u_i)^2}}_{S3_{ii}} + LF + L_{ext} + \underbrace{2\nu \frac{d}{dr}\overline{(\delta u_i)^2}}_{dS2_{ii}} = \frac{4}{3}\varepsilon r \tag{9}$$

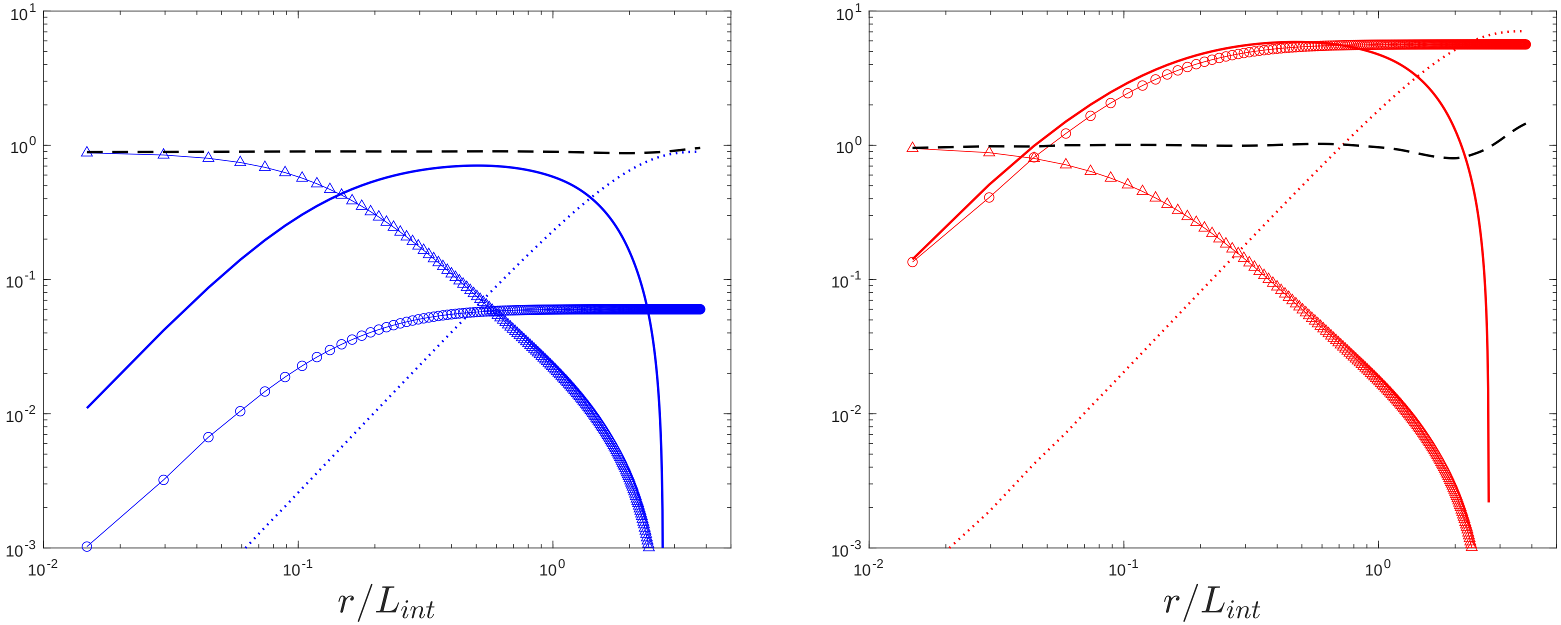


Figure: The case $\rho^n/\rho^s = 10$. (—-) is the convection term $-S3_{ii}^{n,s}$, ($\circ$) the mutual friction term $LF^{n,s}$, ($-\triangle$) the viscous term $dS2_{ii}^{n,s}$, ($\cdot\cdot\cdot$) the external forcing term $L_{ext}^{n,s}$. All terms are normalised by $4/3r\varepsilon^{n,s}$, ($--$) marks the sum of all the terms. (left column) for the normal fluid, (right column) for the superfluid, L_{int} is the integral scale for the total fluid.

The scale-by-scale energy budget

$$\underbrace{-\overline{\delta u_{\|}(\delta u_i)^2}}_{S3_{ii}} + LF + L_{ext} + \underbrace{2\nu \frac{d}{dr}\overline{(\delta u_i)^2}}_{dS2_{ii}} = \frac{4}{3}\varepsilon r \qquad (10)$$

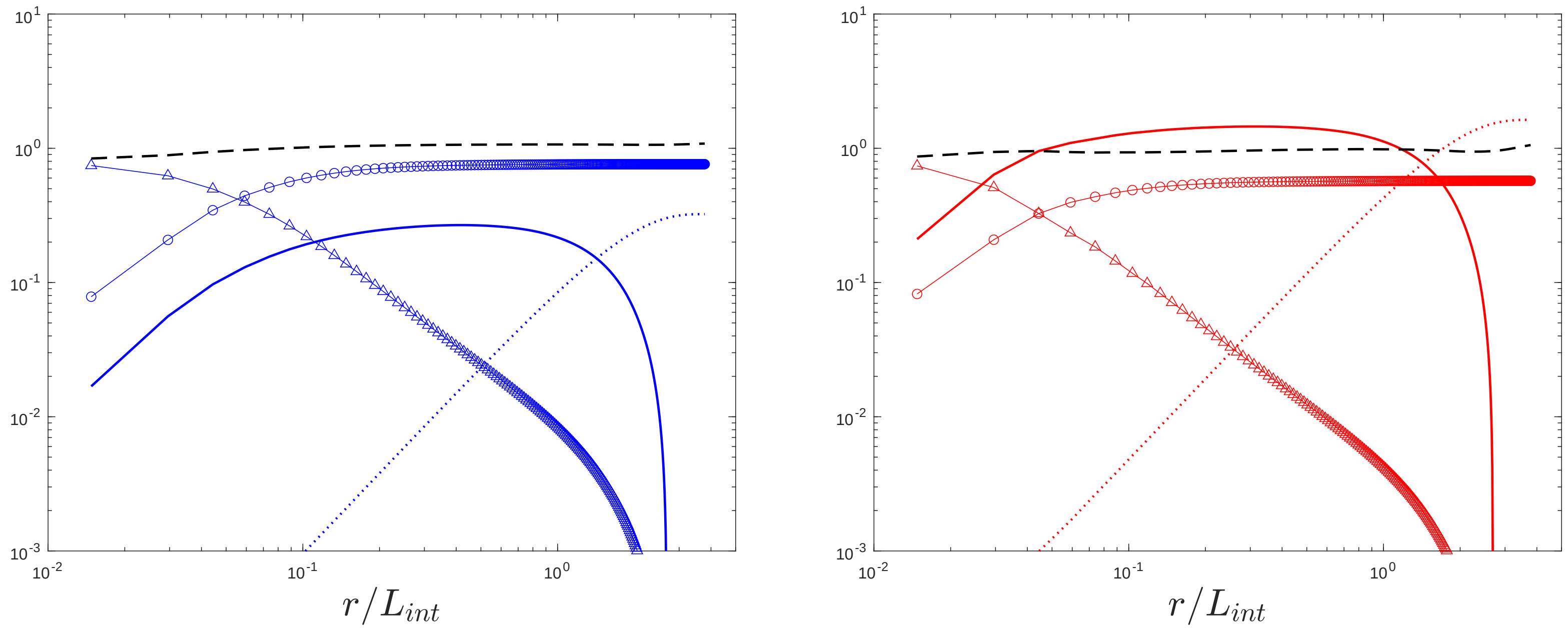


Figure: The case $\rho^n/\rho^s = 0.1$. (—-) is the convection term $-S3_{ii}^{n,s}$, ($\circ$) the mutual friction term $LF^{n,s}$, ($-\triangle$) the viscous term $dS2_{ii}^{n,s}$, ($\cdots$) the external forcing term $L_{ext}^{n,s}$. All terms are normalised by $4/3r\varepsilon^{n,s}$, ($--$) marks the sum of all the terms. (left column) for the normal fluid, (right column) for the superfluid, L_{int} is the integral scale of the total fluid.

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \qquad (11)$$

with $\partial_r \equiv \partial/\partial r$,

$$\begin{aligned} D_{111} &= \overline{(\delta u)^3}; \\ D_{1111} &= \overline{(\delta u)^4}; \\ D_{1122} &= \overline{(\delta u)^2(\delta v)^2}; \\ C(r,t) &= -\frac{4}{r^2} D_{111}(r,t) + \frac{4}{r}\partial_r D_{111} + \partial_r\partial_r D_{111}; \\ Z_{111} &= \overline{3\delta u \left[\left(\frac{\partial u}{\partial x_l}\right)^2 + \left(\frac{\partial u'}{\partial x'_l}\right)^2\right]}, \end{aligned} \qquad (12)$$

where $\delta u = u(x+r) - u(x)$ is the longitudinal velocity increment, $\delta v = v(x+r) - v(x)$ is the transverse velocity increment and double indices indicate summation and a prime denotes variables at point $x + r$. Finally,

$$T_{111} = 3\overline{(\delta u)^2\, \delta\left(\frac{\partial p}{\partial x}\right)}. \qquad (13)$$

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \qquad (14)$$

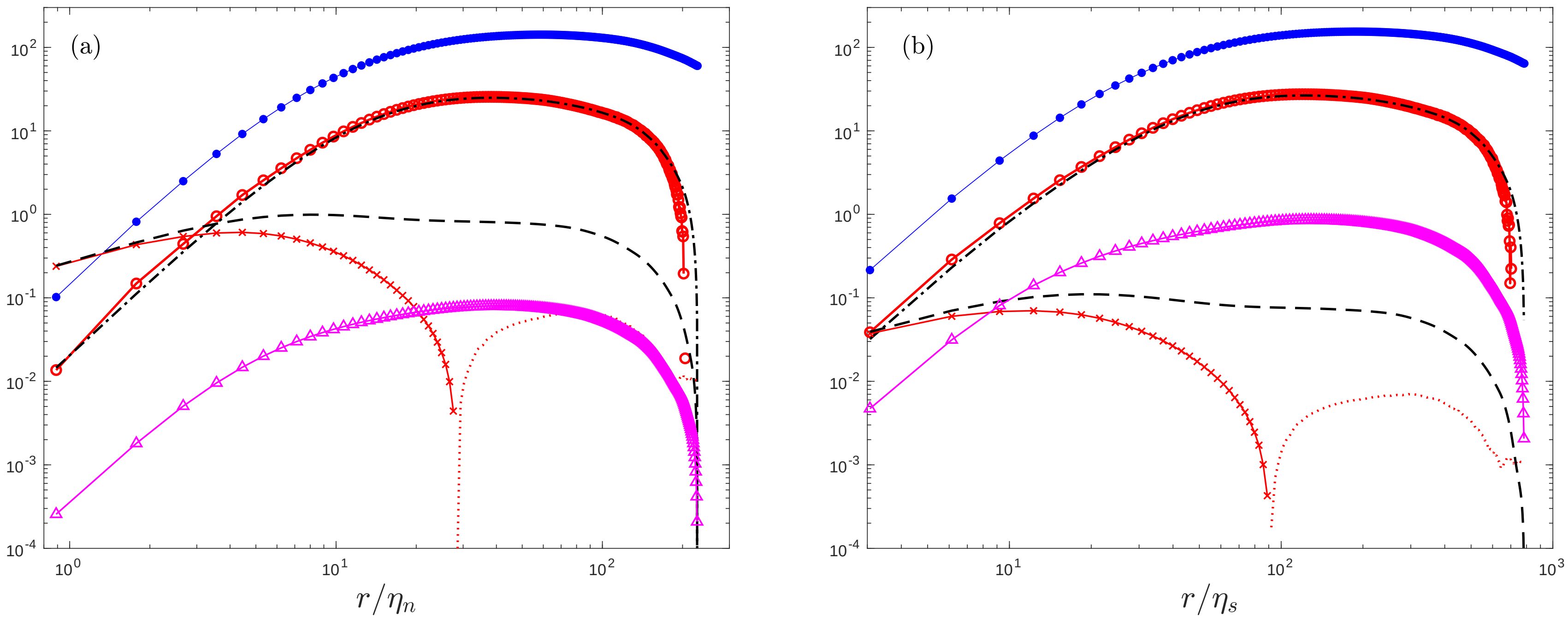


Figure: Balances of different terms in equations for the normal fluid (left) and the superfluid (right) for density ratios $\rho_n/\rho = 0.91$. ($\bullet-$) $(\partial_r + 2/r)D1111$, ($\circ-$) $(\partial_r + 2/r)D1111 - 6/r.D1122$, (black $-\cdot$) $-T111$, ($\times-$) $-2\nu C$ ($\cdot\cdot\cdot$) positive part of $2\nu C$, (- -) $-2\nu Z111$, ($\triangle-$) coupling terms. The plots are dimensionless.

The intermittency of small scales

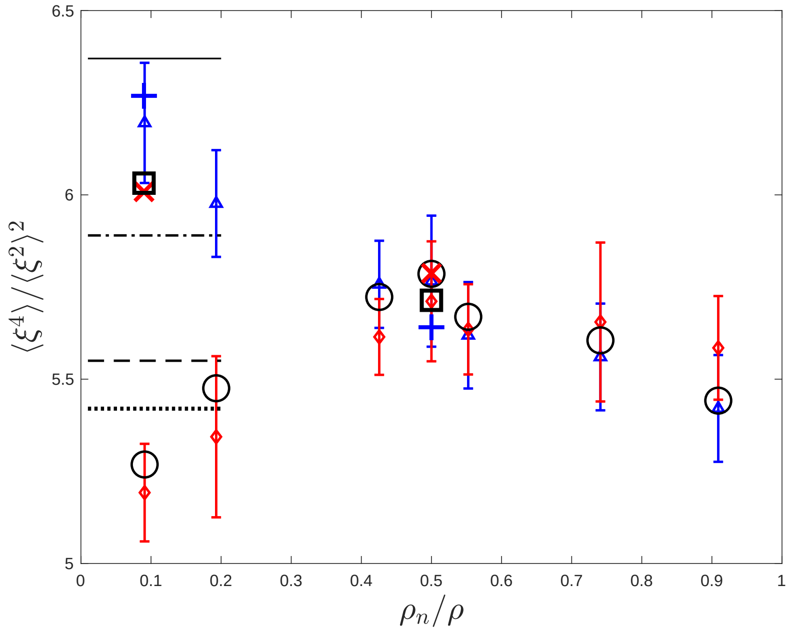


Figure: Flatness factors of the longitudinal velocity gradient $\xi = \partial_x u$ versus density ratio ρ_n/ρ for the normal fluid ($\triangle$), the superfluid ($\diamond$) and total fluid ($\bigcirc$). Error bars are the root-mean-square value of the variance of the flatness factors computed with 20 to 50 snapshots, and 10^8 data points for each snapshot. Horizontal lines mark the flatness factor computed from DNS of classical turbulence (Ishihara, 2007): ($\cdot$ $\cdot$) $Re_\lambda = 94.6$, (- -) $R_\lambda = 94.4$, (- $\cdot$) $R_\lambda = 167$. (–) $R_\lambda = 173$. All points are computed for $N = 512$, except the following ones, based on $N = 1024$ resolution: big blue $+$ (normal fluid), big red $\times$ (superfluid) and big black $\square$ (total fluid).

Why add the PNAS 2025 results?

[redacted]

The previous slides describe mutual-friction exchanges in the two HVBK components separately.
The new result is to write the scale-by-scale budget for the **He-II mixture**.

Question

- At which scales does mutual friction dissipate energy?
- Does mutual friction change inertial-range intermittency?
- How should a Reynolds number be defined for a two-fluid system?

[redacted]

- The two components exchange intense momentum locally.
- For the **mixture**, frictional dissipation is sub-dominant and mainly far-dissipative.
- Apparent temperature dependence of intermittency is largely a finite-Reynolds-number/dissipative-scale effect.

Key message: He-II inertial-range statistics remain remarkably classical when the proper effective scales are

A Kármán–Howarth budget for He-II as a mixture

Classical HIT: one-fluid reference

$$\frac{4}{3}\varepsilon r = -S_3(r) + 2\nu \frac{dS_2}{dr}.$$

He-II mixture: mass-weighted two-fluid budget

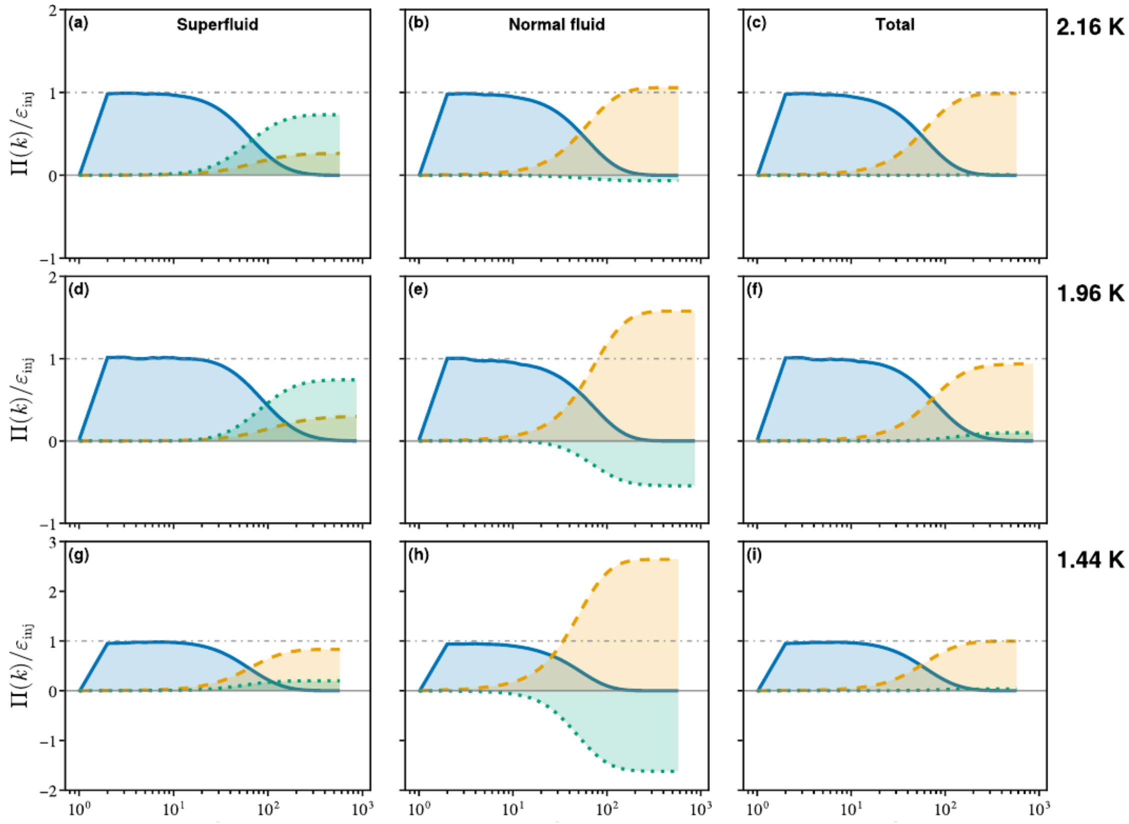
$$\frac{4}{3}\widetilde{\varepsilon}_{\mathrm{inj}}\, r = -\widetilde{S}_3(r) + 2\widetilde{\nu}\frac{d\widetilde{S}_2}{dr} + \varphi_{ns}(r),$$

$$\varphi_{ns}(r) = \frac{2\rho_n\rho_s}{\rho^2 r^2}\int_0^r C_{ns}(r')\, r'^2\, dr', \qquad C_{ns} = \langle \boldsymbol{v}_{ns}(\boldsymbol{x}+\boldsymbol{r})\cdot \boldsymbol{F}_{ns}(\boldsymbol{x}) + \boldsymbol{v}_{ns}(\boldsymbol{x})\cdot \boldsymbol{F}_{ns}(\boldsymbol{x}+\boldsymbol{r})\rangle.$$

- The forcing term is the total injected energy of the mixture.
- $\widetilde{S}_3$ and $\widetilde{S}_2$ are mass- and viscosity-weighted two-fluid structure functions.
- $\varphi_{ns}(r)$ is the scale contribution of mutual friction.
- In the limit $r \to 0$: $\widetilde{\varepsilon}_{\mathrm{inj}} = \widetilde{\varepsilon}_\nu + \varepsilon_{ns}$.

Polanco et al., PNAS 122, e2426598122 (2025).

Energy fluxes: exchanges are intense, but total frictional dissipation is small



Reading the figure

- Blue: cascade flux $\Pi(k)$.
- Orange: cumulative viscous dissipation $D(k)$.
- Green: inter-component flux / cumulative mutual-friction dissipation.

For each component, mutual friction is a strong energy-exchange mechanism at small scales.

For the **mixture**, the total frictional contribution remains modest compared with viscous dissipation and is confined to far-dissipative scales.

Fig. 1 from Polanco et al., PNAS 122, e2426598122 (2025). Simulations: $T \simeq 2.16, 1.96, 1.44$ K.

Effective microscales: disentangling temperature and Reynolds number

A unique Taylor microscale for the He-II mixture

$$\widetilde{\varepsilon}_\nu = 15\widetilde{\nu}\frac{v_{\rm rms}^2}{\widetilde{\lambda}^2}, \qquad \frac{1}{\widetilde{\lambda}^2} = \frac{\rho_n\nu_n/\lambda_n^2 + \rho_s\nu_s/\lambda_s^2}{\rho_n\nu_n + \rho_s\nu_s}, \qquad \widetilde{R}_\lambda = \frac{v_{\rm rms}\widetilde{\lambda}}{\widetilde{\nu}}.$$

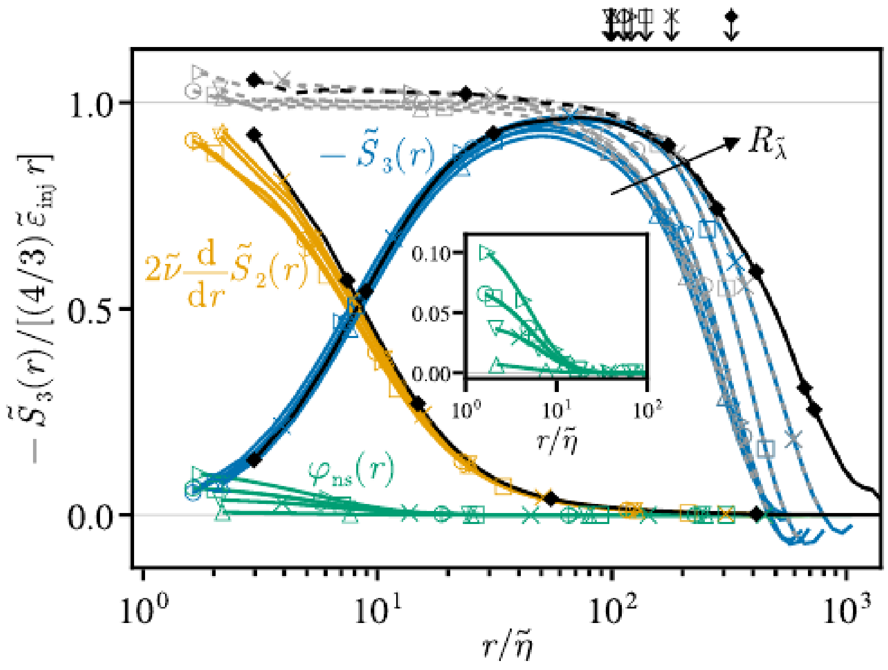
An effective Kolmogorov scale

$$\widetilde{\eta} = \left(\frac{\nu_{\rm tot}^3}{\widetilde{\varepsilon}_{\rm inj}}\right)^{1/4}, \qquad \nu_{\rm tot} = \widetilde{\nu}\frac{\widetilde{\varepsilon}_{\rm inj}}{\widetilde{\varepsilon}_\nu}.$$

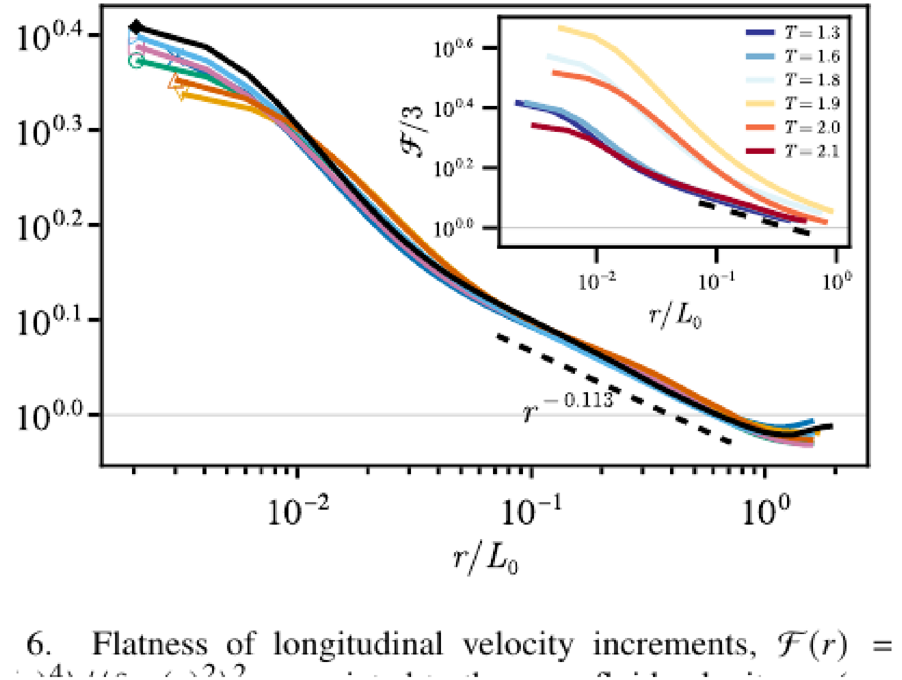
- The definition is based on the exact mixture energy budget.
- It avoids comparing the normal and superfluid components using incompatible single-component Reynolds numbers.
- It separates temperature effects from finite-Reynolds-number effects.
- It gives the correct normalization for inertial- and dissipative-scale statistics.

Polanco et al., PNAS 122, e2426598122 (2025).

Scale-by-scale budget and intermittency



Mixture budget normalized by $\frac{4}{3}\widetilde{\varepsilon}_{\rm inj}r$ and $r/\widetilde{\eta}$.



Flatness $\mathcal{F}(r) = \langle \delta v_{||}^{\mathbf{4}} \rangle / \langle \delta v_{||}^{\mathbf{2}} \rangle^{\mathbf{2}}$.

- The mixture budget closes well over the resolved scales; the friction term is visible but sub-dominant.
- When normalized with the effective scales, He-II flatness follows the classical intermittency trend in the cascade range.
- The apparent temperature effect occurs mainly in the dissipative range, hence it should be interpreted as a finite-Reynolds-number/scale-normalization effect.

Figs. 5–6 from Polanco et al., PNAS 122, e2426598122 (2025).

Updated take-home message for the AIMS talk

What remains from the previous HVBK study

- Mutual friction produces intermittent, localized exchanges between the normal and superfluid components.
- These exchanges are tied to intense vorticity/enstrophy regions.
- Component-by-component budgets reveal strong redistribution of energy.

What the PNAS 2025 paper adds

- The relevant inertial-range object is the **two-fluid mixture**.
- A He-II Kármán–Howarth equation defines effective microscales and an effective Reynolds number.
- Inertial-range statistics are classical-like; temperature effects mainly affect dissipative scales.
- The inter-vortex distance follows $\ell/L_0 \simeq 0.5 Re_\kappa^{-3/4}$.

Conclusion: mutual friction maintains the joint cascade, but does not redefine inertial-range intermittency.

Conclusions for AIMS

1. Component level

Mutual friction produces localized, intermittent energy exchanges concentrated around intense vorticity/enstrophy regions.

2. Mixture level

The two-fluid mixture obeys a Kármán-Howarth-type scale-by-scale budget; net frictional dissipation is small and mainly far-dissipative.

3. Statistics

Effective Reynolds and Kolmogorov scales disentangle temperature from Reynolds-number effects; cascade-range intermittency remains classical-like.

Final take-home: He-II turbulence is quantum in its microscopic vorticity, but its properly normalized inertial-range energy cascade is remarkably classical.

Publications and discussion

Publications supporting this talk

Zhang, Danaila, Lévêque & Danaila, J. Fluid Mech. 962, A22 (2023): component-level higher-order budgets and frictional exchanges.

Polanco, Roche, Danaila & Lévêque, PNAS 122, e2426598122 (2025): mixture energy budget, effective microscales and intermittency reinterpretation.

Suggested discussion points

How far can the HVBK mixture budget be pushed toward LES closures?

Can the effective Reynolds number be used to compare experiments, DNS and models systematically?

Which scale-by-scale diagnostics should be exported to atmospheric/finite-Re turbulence?

Thank you