# Dynamical Reductions for Solitonic Filaments

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- Mason Porter (UCLA)
- Yuri Kivshar (Camberra)
- Pedro Torres (Granada)
- Todd Kapitula (Calvin Col.)
- Keith Promislow (MSU)
- Yannis Kevrekidis (Johns Hopkins
- Lincoln Carr (Col. Sch. Mines)
- etc, etc, etc ...

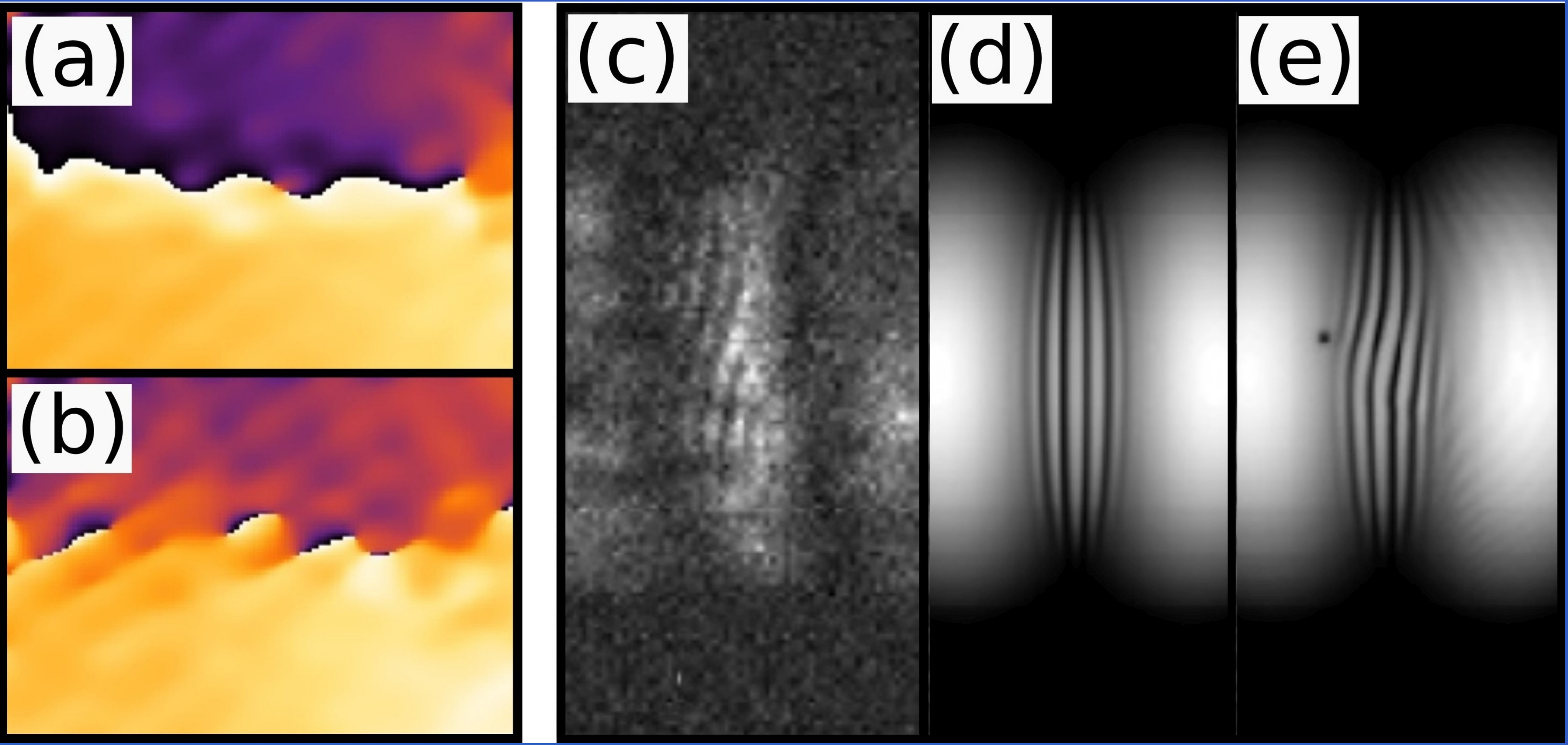
# Road map

- Introduction
  - Motivation: BEC physical experiments
- Single solitonic filaments in NLS/BECs:
  - Adiabatic Invariant (AI) methodology
  - Soliton stripes
  - Ring dark soliton (RDS) [brief]
  - Dark-Bright solitons [brief]
- Other filament reductions:
  - Filament-filament interactions in NLS (AI+VA) [brief]
  - Filament-vortex interactions in NLS (VA)
- Conclusions
  - Recap (inc. list of other filament reductions)
  - Outlook

# Solitonic filaments

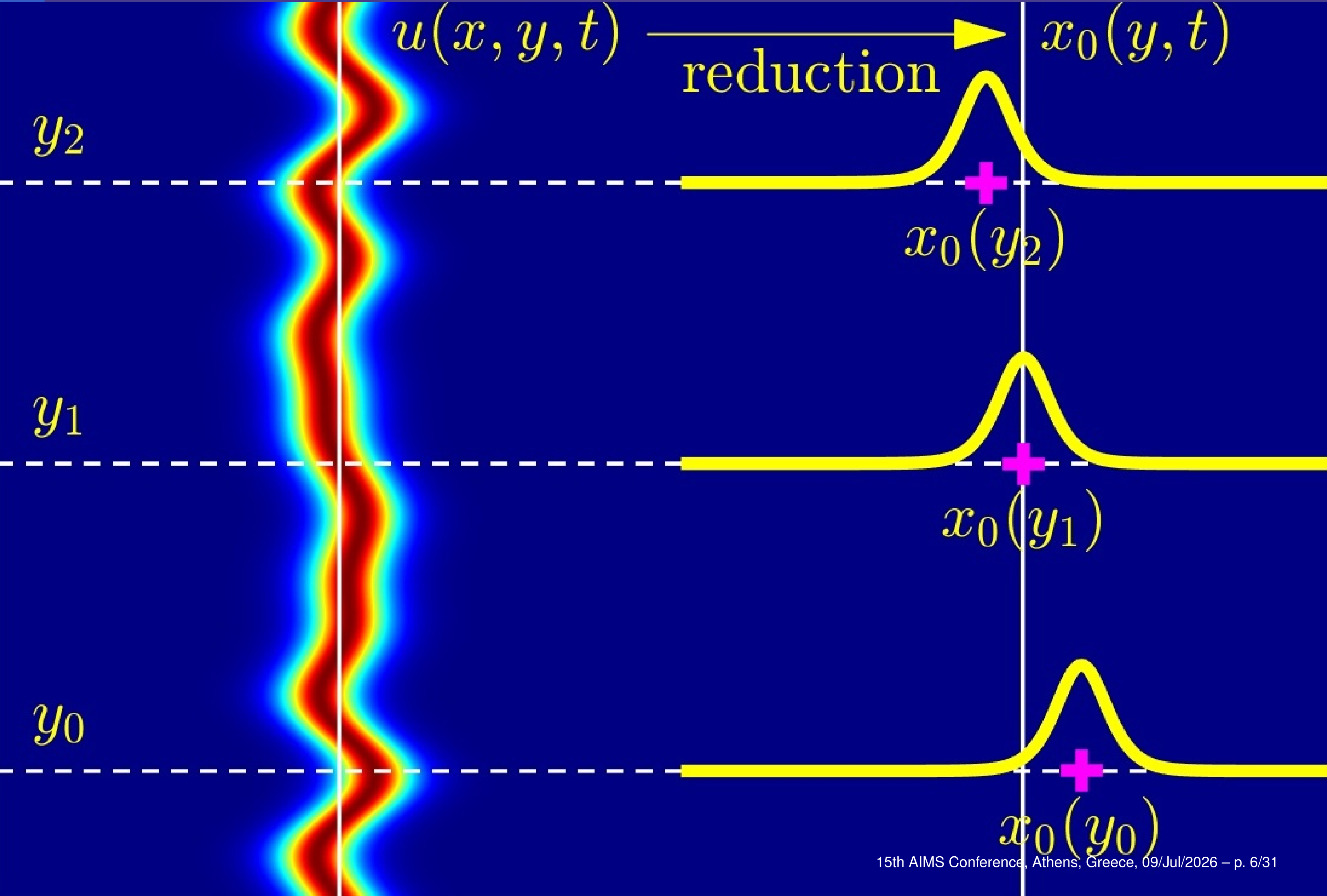
# in NLS-type systems

# Solitonic filaments: experimental motivation




- (a) & (b): Exciton-polariton BEC, Daniele Sanvitto, Lecce.
- (c)–(e): DSSs-vortex in atomic BEC, Brian Anderson, Arizona.

# Dynamical reduction for filaments in a picture:



# Reduction approach (AI+VA+EL+pert.+Galerkin+...)

- Want to solve for dynamics of a stripe in 2D: $\boxed{\mathrm{PDE}[u(x,y,t)] = 0}$

  $\Rightarrow$ PROJECT PDE onto a lower-dim. ansatz ($\rightarrow$ PDE/ODE) !

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$$\mathrm{PDE}[u(x,t)]=0 \quad \Leftrightarrow \quad H_{1\mathrm{D}}=\int_{-\infty}^{\infty}\mathcal{H}_{1\mathrm{D}}(u)\,dx=\mathrm{const.}$$

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- Build a 2D stripe: $u_{2\mathrm{D}}(x,y,t)=u_{1\mathrm{D}}(x-x_0(y,t))$ ($\Rightarrow |u_y|=|x_{0y}|\cdot|u_x|$).
- Evaluate $H_{2\mathrm{D}}$ on the stripe: $H_{2\mathrm{D}}=\iint_{-\infty}^{\infty}\mathcal{H}_{2\mathrm{D}}(u_{1\mathrm{D}}(x-x_0(y,t)))\,dx\,dy$.

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- Massage/blend/simmer ($\int dx$, by parts, EL, ...) and $dH/dt=0$ $\Rightarrow$

$\boxed{\mathrm{PDE}[x_0(y,t)]=0}$ $\leftarrow$ Lower dimensional than original !!!

# Adiabatic Invariant (AI) Approach : 1D : dark soliton

- 1D NLS for external potential $V$:

$$iu_t = -\frac{1}{2}u_{xx} + |u|^2 u + V(x)\, u.$$

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  Hamiltonian: $H_{1\mathrm{D}} = \frac{1}{2}\int_{-\infty}^{\infty}\left[|u_x|^2 + \left(\mu - |u|^2\right)^2\right]dx.$

  Dark soliton: $u_{\mathrm{ds}} = e^{-i\mu t}\left[\sqrt{\mu - v^2}\,\tanh\left(\sqrt{\mu - v^2}(x - x_0)\right) + iv\right]$ :

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- Bring back $V \rightarrow$ AI: consider background *slowly* varying ($u_{xx} \sim 0$)
  $\rightarrow$ steady state: $u(x,t) = w(x)e^{-i\mu t} \Rightarrow \mu w \sim w^3 + V\,w \Rightarrow w^2 \sim \mu - V$.

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- Thus background $w^2 \sim \mu - V$ is captured with $\mu \rightarrow \mu - V(x)$:

$$H_{1\mathrm{D}} \simeq \frac{4}{3}\left(\mu - V(x_0) - \dot{x}_0^2\right)^{3/2} \;\Rightarrow\; \ddot{x}_0 \simeq -\frac{1}{2}V'(x_0),$$

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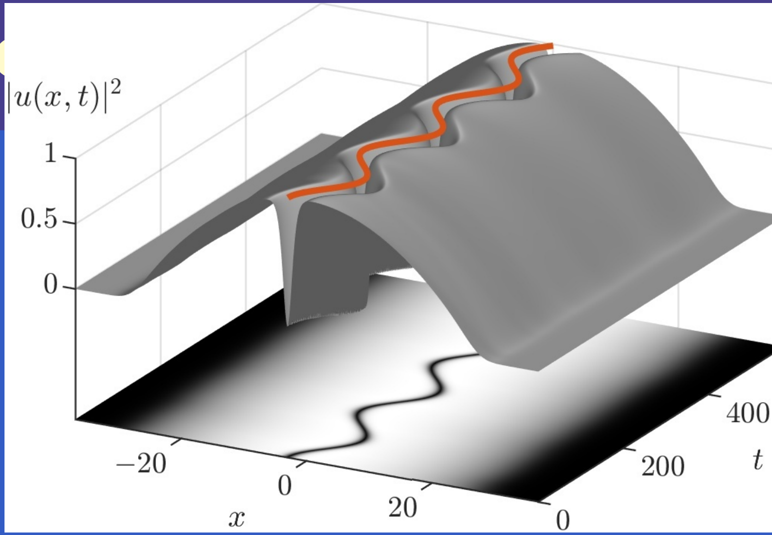


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- Reduced filament PDE : $dH/dt = 0 \Rightarrow$ PDE for $x_0(y,t)$:

$$\boxed{B\, x_{0tt} + \frac{A}{3} x_{0yy} = x_{0y} x_{0t} x_{0yt} - \frac{1}{2} V'(x_0)\left(B - x_{0y}^2\right)}$$

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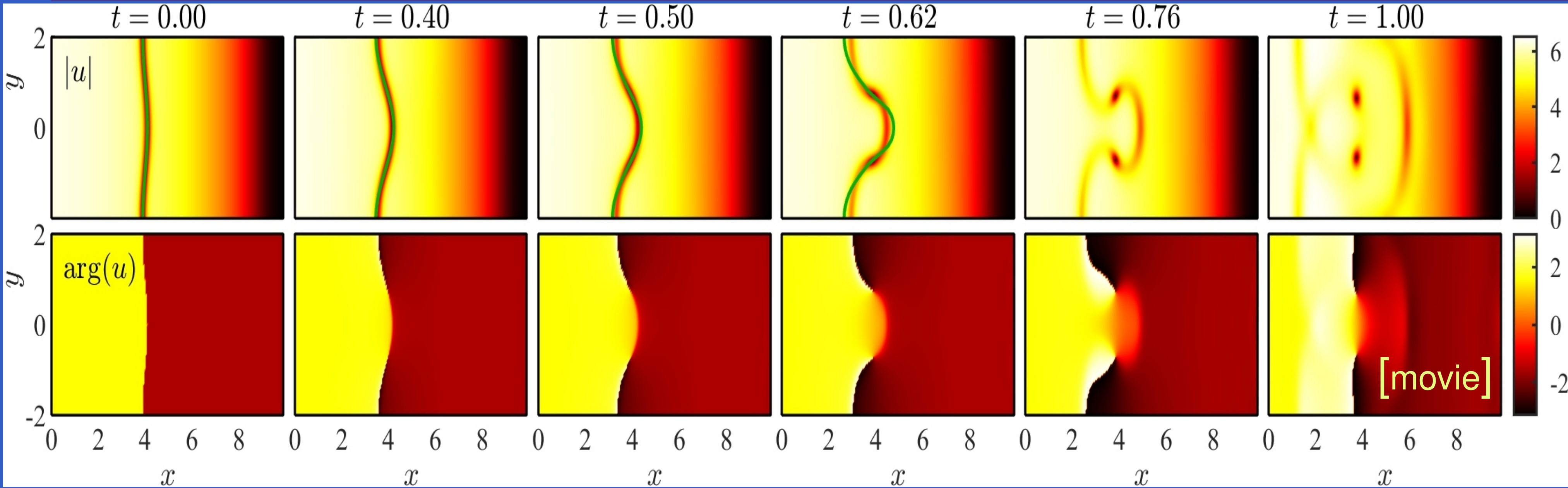
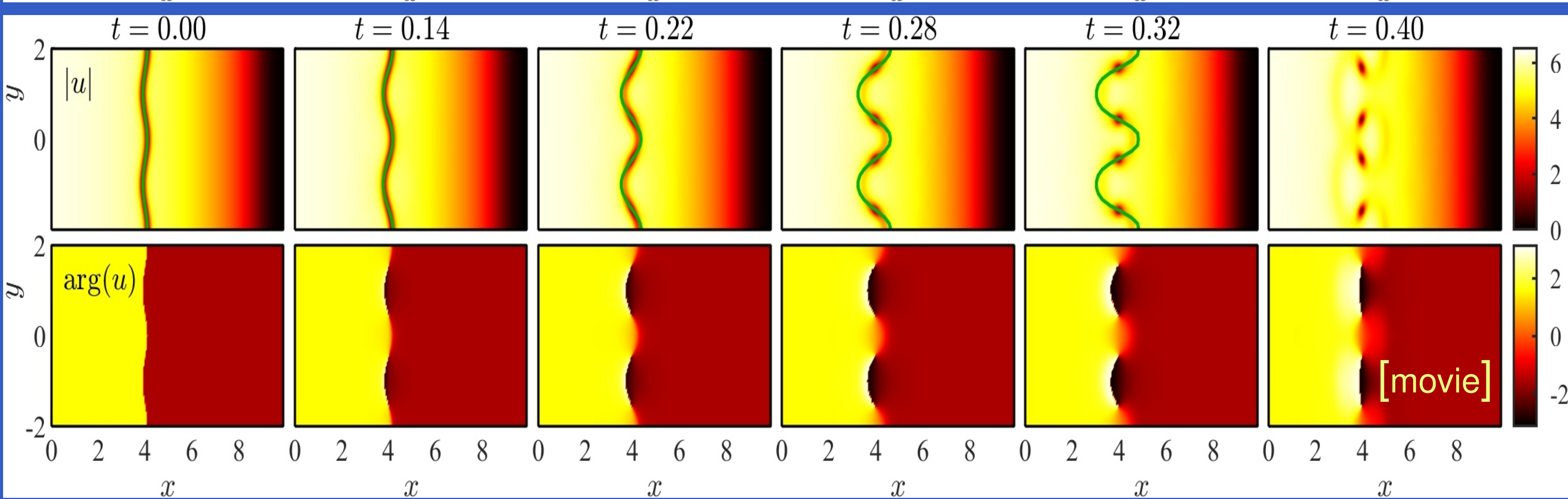
- At linear level with $V = 0$ one recovers: $x_{0tt} + \frac{\mu - v^2}{3}x_{0yy} = 0$.

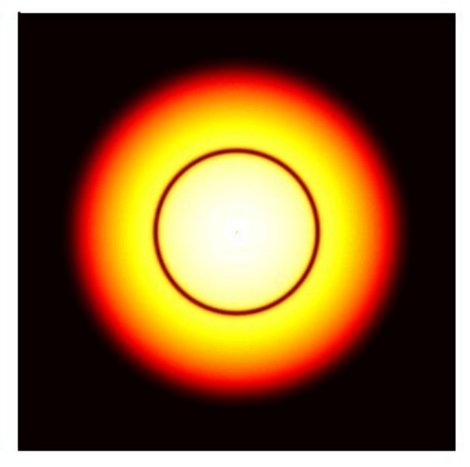
# DS snaking for $V \neq 0$ : $V(x,y) = \frac{1}{2}\Omega^2 x^2$ : PDE vs. AI



# AI : Ring dark soliton (RDS) $\rightarrow$ dynamics

- Start with NO azimuthal perturbations, i.e. radius given by $R = R(t)$. [Kamchatnov+Korneev PLA'10 and Konotop+Pitaevskii PRL'04] :



$$E = 2\pi R(\mu - V(R) - \dot{R}^2)^{3/2}$$

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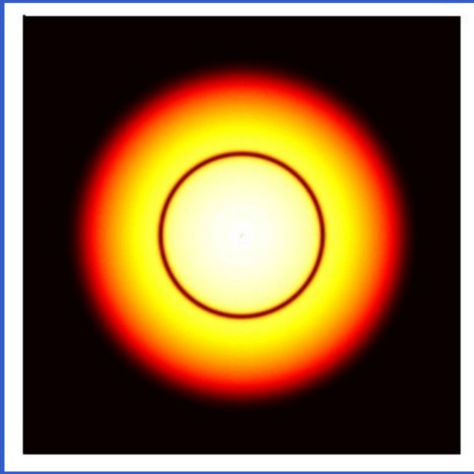

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- Generalize to RDS filament : $R(t) \to R(\theta, t)$. Energy:

$$E = \int_0^{2\pi} R \left(1 + \frac{R_\theta^2}{2R^2}\right) \left(\mu - V(R) - R_t^2\right)^{3/2} d\theta.$$

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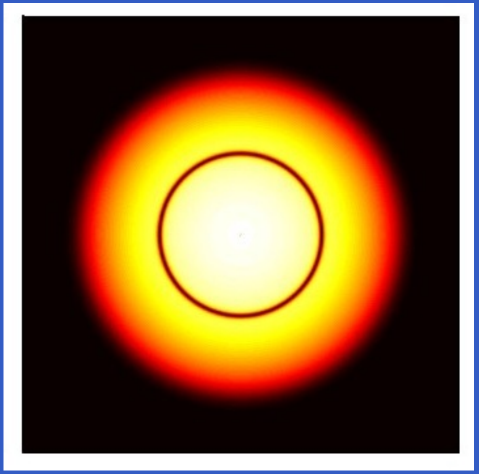

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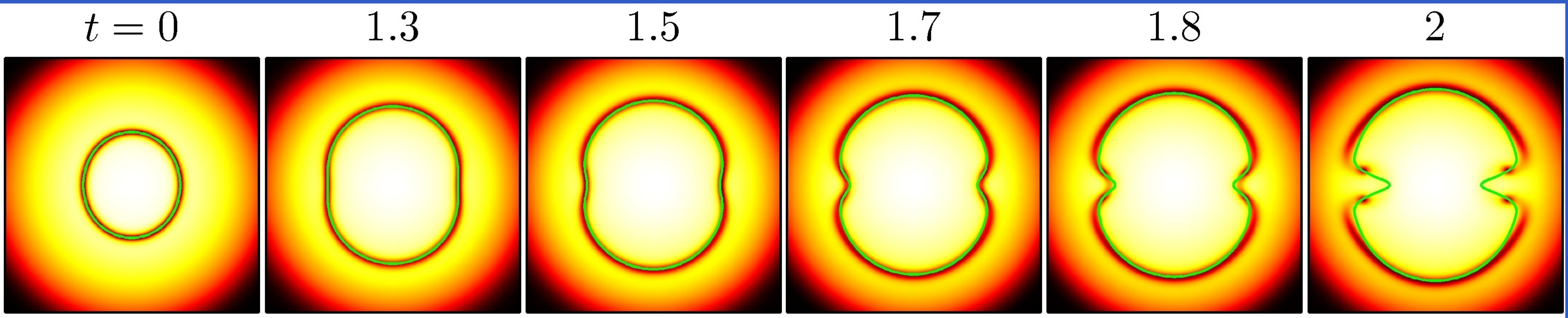
- Apply the AI methodology ($dE/dt = 0$) $\rightarrow$ PDE for $R(\theta, t)$:

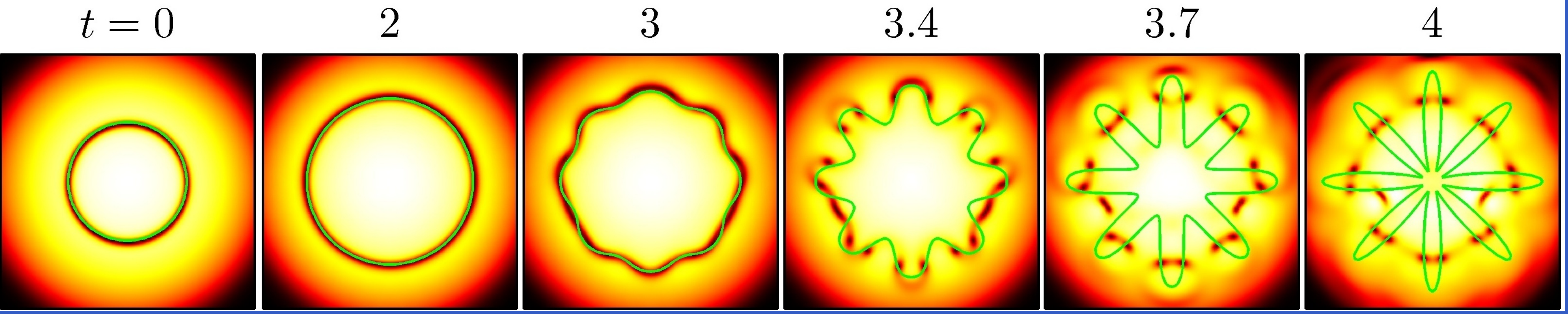
$$\boxed{CD - \frac{R_{\theta\theta}}{R}C = -\frac{R_\theta}{R}\left(\frac{3}{2}V'(R)R_\theta + 3R_tR_{t\theta}\right) + RD\left(\frac{3}{2}V'(R) + 3R_{tt}\right)}$$

with $C \equiv \mu - V(R) - R_t^2$ and $D \equiv 1 + \frac{R_\theta^2}{2R^2}$.

# Ring dark soliton : PDE vs AI




[movie]




[movie]

# AI : Dark-bright stripes with $V \neq 0$

- 2-component GPE:

$$\begin{aligned} iu_t &= -u_{xx}/2 + \left[V_d + |u|^2 + |v|^2 - \mu_d\right] u, \\ iv_t &= -v_{xx}/2 + \left[V_b + |u|^2 + |v|^2 - \mu_b\right] v. \end{aligned}$$

- Dark-Bright (DB) soliton when $V = 0$:

$$\begin{aligned} u_{\rm ds} &= \sqrt{\mu_d}\,\left[\cos(\alpha)\tanh(\nu(x - x_0)) + i\sin(\alpha)\right], \\ v_{\rm bs} &= \sqrt{N_b\nu/2}\,\mathrm{sech}(\nu(x - x_0))e^{-i\mu_b t}e^{i\,\dot{x}_0 x}. \end{aligned}$$

- 1D energy density: $G_{\rm DB,1D} = \frac{4}{3}\mathcal{A}^3 - 2\dot{x}_0{}^2\mathcal{A} + N_b\left(V_b - \frac{1}{2}V_d\right)$, where $\mathcal{A} = \mathcal{A}(x) = [\mu_d - V_d(x) + N_b^2/16]^{1/2}$.

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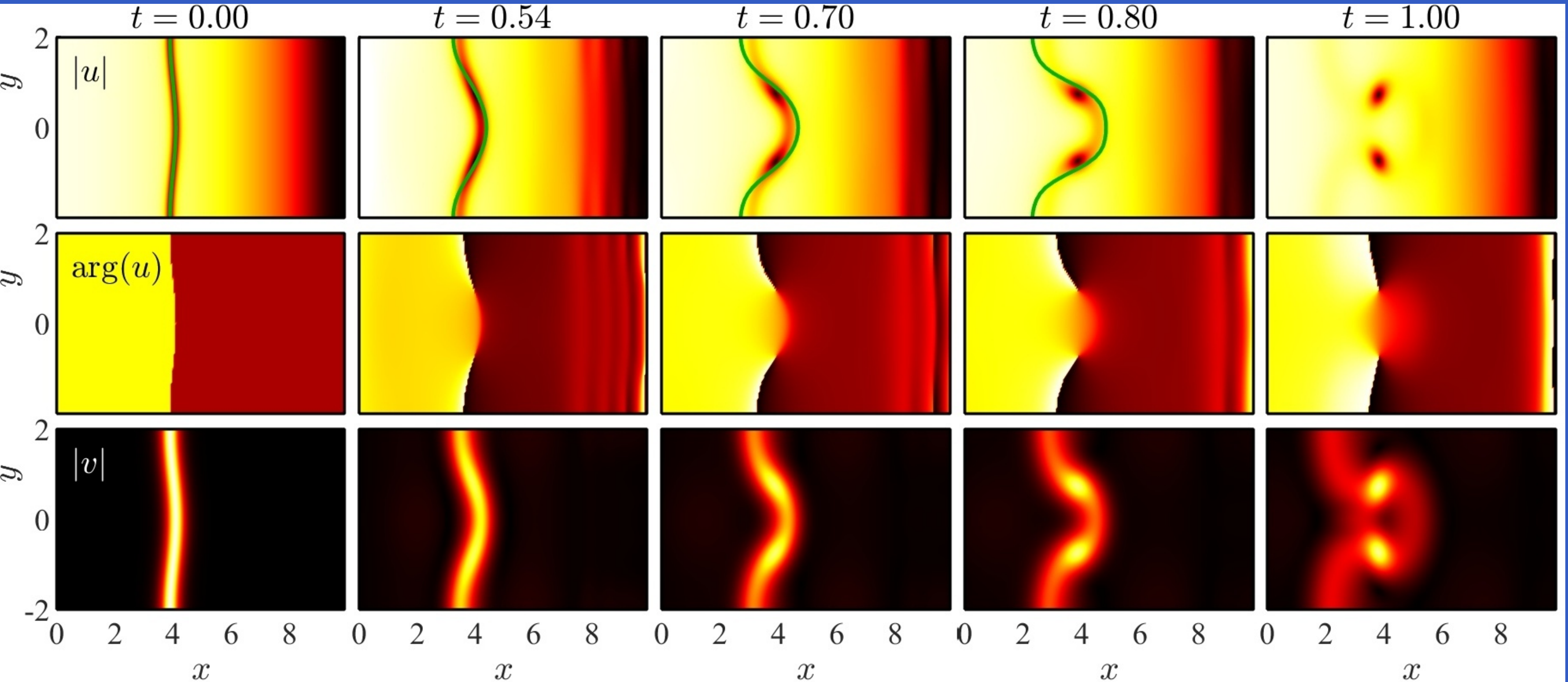
- 2D DB stripe: $u = u_{\rm ds}(x - x_0(y,t))$ and $v_{\rm bs} = v(x - x_0(y,t))$:
- 2D energy density + AI when $V \neq 0$ : $\mu \to \mu - V$ :

$$G_{\rm DB,2D} = \int G_{\rm DB,1D} + (x_{0y})^2\left(\frac{2}{3}\mathcal{A}^3 - \frac{N_b^2\mathcal{A}}{8} + \frac{N_b^3}{48} - (x_{0t})^2\,\frac{\mu_d - V_d + \frac{N_b^2}{8}}{\mathcal{A}}\right) dy,$$

- $\to$ PDE for the DB: a bit messy (not shown here).

# Dark-bright stripes : PDE vs AI




[movie]

[Also: stability spectrum nicely captured by linearization of AI PDE.]

# Filament-filament interactions

# AI : Interacting DS stripes in NLS

- 1 DS: $u = e^{-i\mu t}\left[\sqrt{\mu - v^2}\tanh\left(\sqrt{\mu - v^2}(x - x_0)\right) + iv\right],$
- 1 DS Hamiltonian: $H = \frac{4}{3}\int_{-\infty}^{\infty} B\, A^{3/2}\, dy,$
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$$E = 2\int_{-\infty}^{\infty}\left(\frac{4}{3}B\, A^{3/2} - 8A^{3/2}e^{-4A^{1/2}x_0}\right)dy.$$

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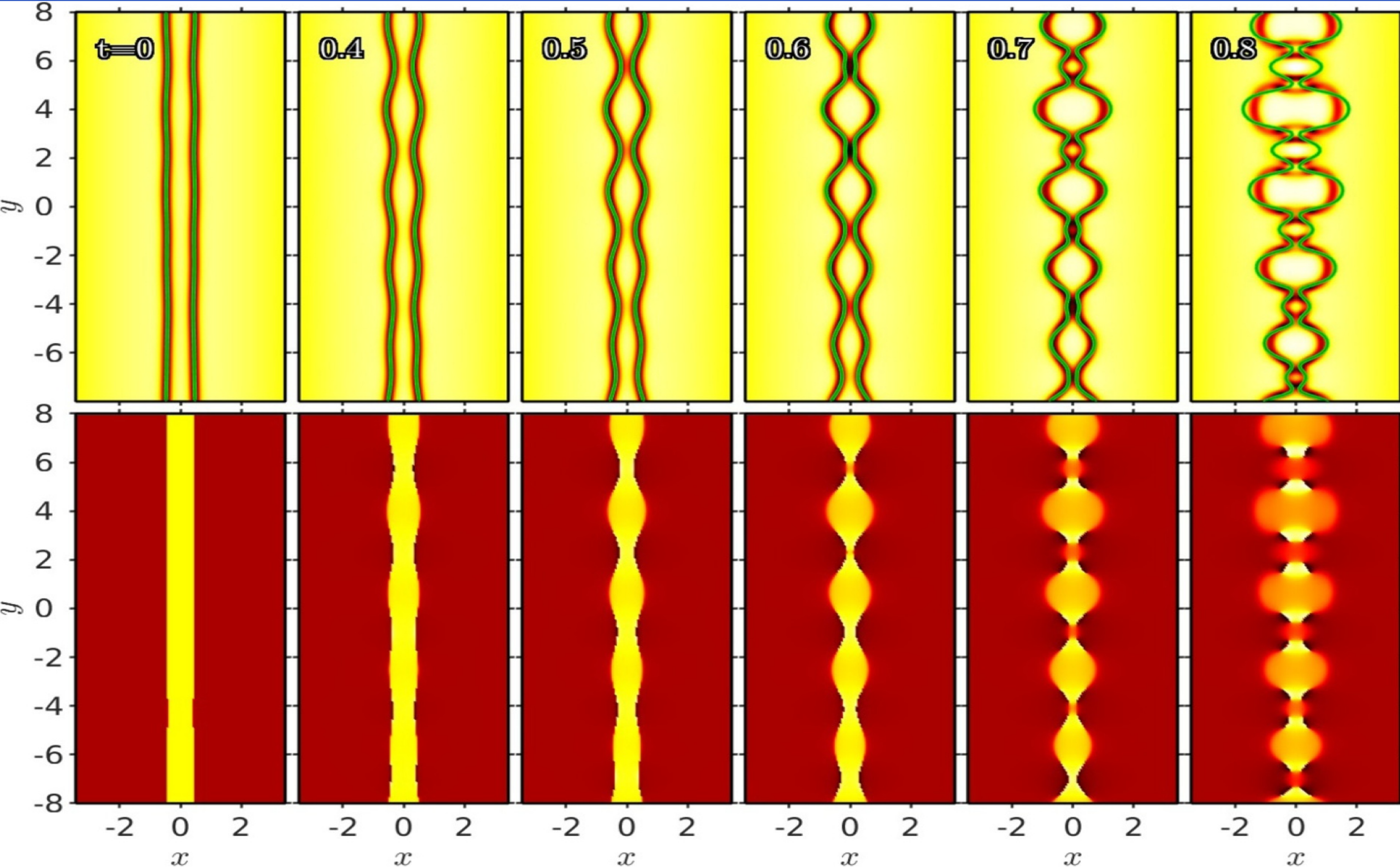
$$E = 2\int_{-\infty}^{\infty}\left(\frac{4}{3}B\, A^{3/2} - 8A^{3/2}e^{-4A^{1/2}x_0}\right) dy.$$

- Reduced PDE for $x_0$ (half DS separation) $\rightarrow$

$$\boxed{\begin{aligned} B\left(x_{0tt} + \frac{V'(x_0)}{2}\right) + \frac{A}{3}x_{0yy} &= \frac{V'(x_0)}{2}{x_{0}}_y^2 + x_{0y}x_{0t}x_{0ty} \\ &\quad - \left[(V'(x_0) + 2x_{0tt})(-3 + 4A^{1/2}x_0) - 8A^{3/2}\right] e^{-4A^{1/2}x_0} \end{aligned}}$$

# Interacting DS stripes in NLS: PDE vs reduction

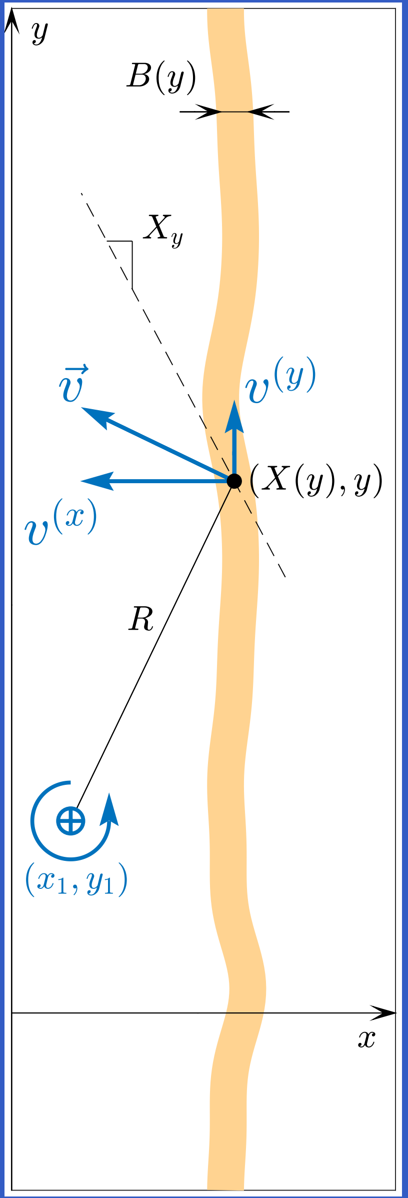


# Filament-vortex interactions

# DSS-vortex interactions in NLS: Variational approach

## (A) VORTEX $\leftrightarrow$ FILAMENT INTERACTION (VA)

- DSS: $u_{\rm dss} = B(y,t)\tanh\left[B(y,t)(x - X(y,t))\right] + i\mathcal{V}(y,t)$, width: $B(y,t)$, horizontal vel.: $\mathcal{V}(y,t)$, with $\mathcal{V}^2 + B^2 = \mu$.
- Point vortex: $u_{\rm v} = \sqrt{\mu}\, e^{iS\theta}$, $\theta = \tan^{-1}[(y-y_1),(x-x_1)]$.



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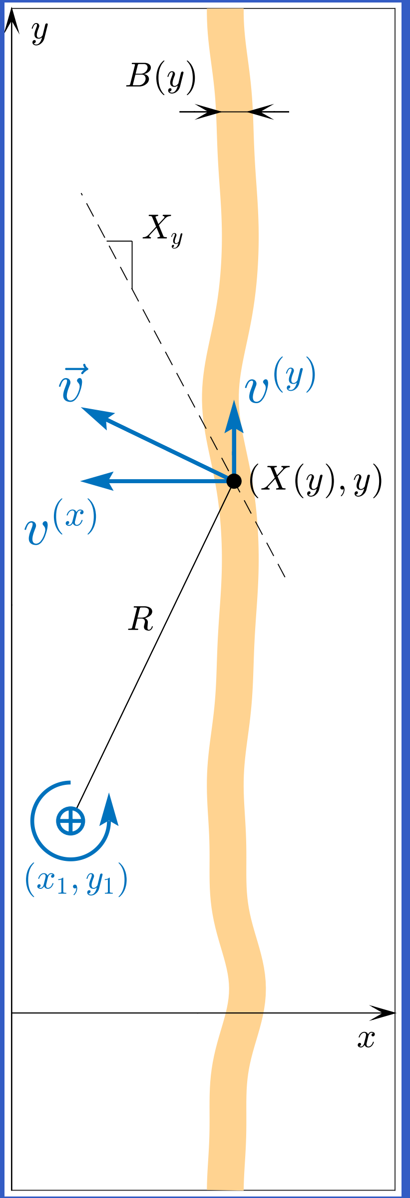
- Point vortex: $u_{\rm v} = \sqrt{\mu}\, e^{iS\theta}$, $\theta = \tan^{-1}[(y-y_1),(x-x_1)]$.
- VA (includes vortex $\rightarrow$ filament):

$$\left\{\begin{array}{rcl} X_t &=& \mathrm{PDE}_{\text{single DSS}}[X,\mathcal{V}] + v^{(x)} - v^{(y)}\,X_y, \\ \mathcal{V}_t &=& \mathrm{PDE}_{\text{single DSS}}[X,\mathcal{V}], \end{array}\right.$$

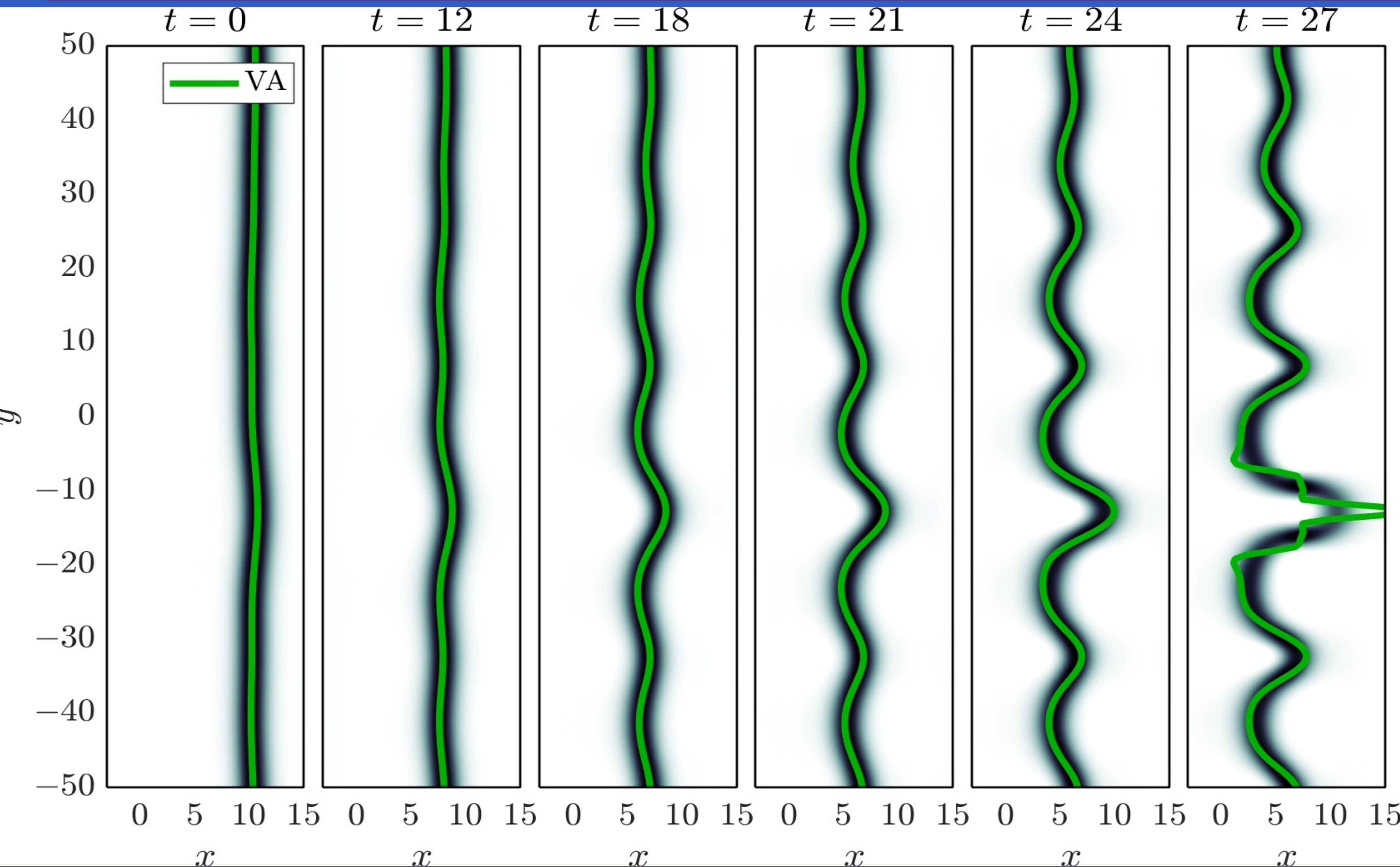
where now

$$v^{(x)} = -S\,\frac{y-y_1}{R^2}, \qquad v^{(y)} = +S\,\frac{x-x_1}{R^2}.$$

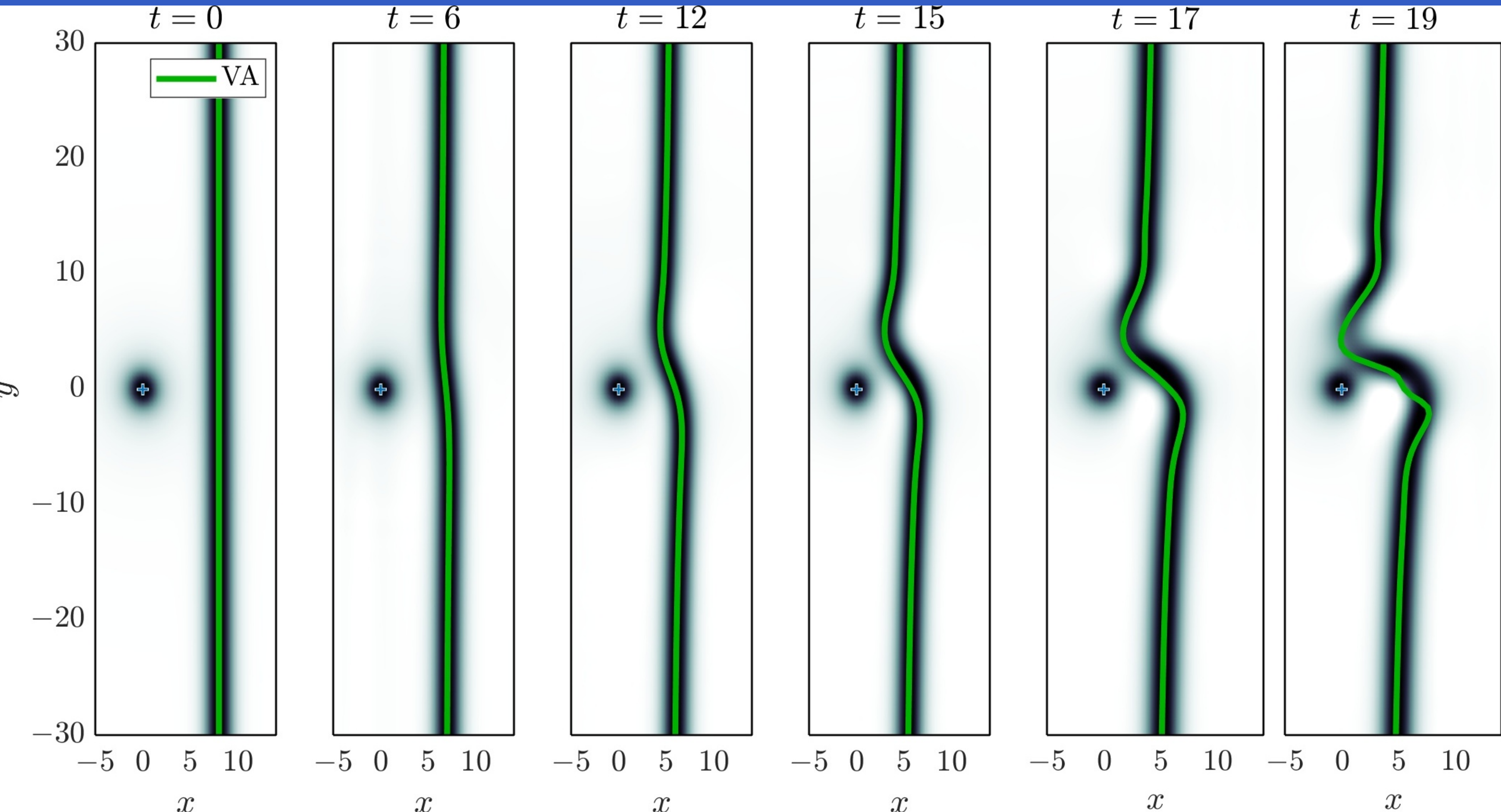
- Includes contribution from filament slope $X_y$ !!!). :)))



# DSS snaking: Variational approach



# vortex → DSS interactions: Variational approach




[movie]

# DSS-vortex interactions in NLS: Variational approach

## (B) VORTEX $\leftrightarrow$ FILAMENT & VORTEX $\leftrightarrow$ VORTEX

- Same methodology vortex $\leftrightarrow$ filament and add vortex $\leftrightarrow$ vortex
- Consider point vortices (NLS $\rightarrow$ inviscid Euler + quantum pressure)

$$\begin{cases} X_t = \mathrm{PDE}_{\text{single DSS}}[X, \mathcal{V}] + v^{(x)} - v^{(y)}\, X_y, \\ \mathcal{V}_t = \mathrm{PDE}_{\text{single DSS}}[X, \mathcal{V}], \end{cases}$$

with $\quad v^{(x)} = -\sum_n S_n \dfrac{y - y_n}{R_n^2}, \qquad v^{(y)} = +\sum_n S_n \dfrac{x - x_n}{R_n^2}.$

# DSS-vortex interactions in NLS: Variational approach

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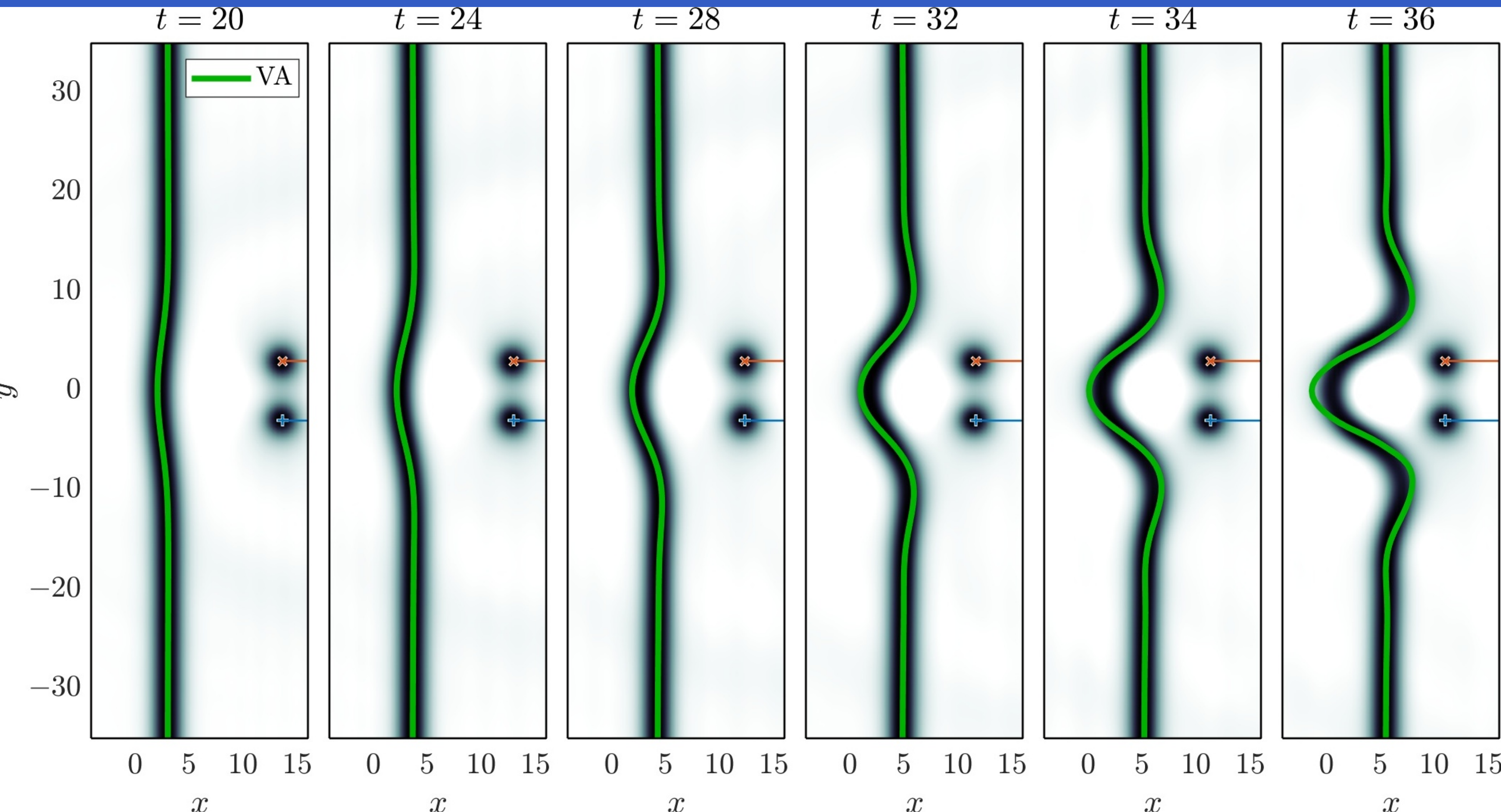
with $\quad v^{(x)} = -\sum_n S_n \frac{y - y_n}{R_n^2}, \qquad v^{(y)} = +\sum_n S_n \frac{x - x_n}{R_n^2}.$

- Superposition of $N$ vortices:

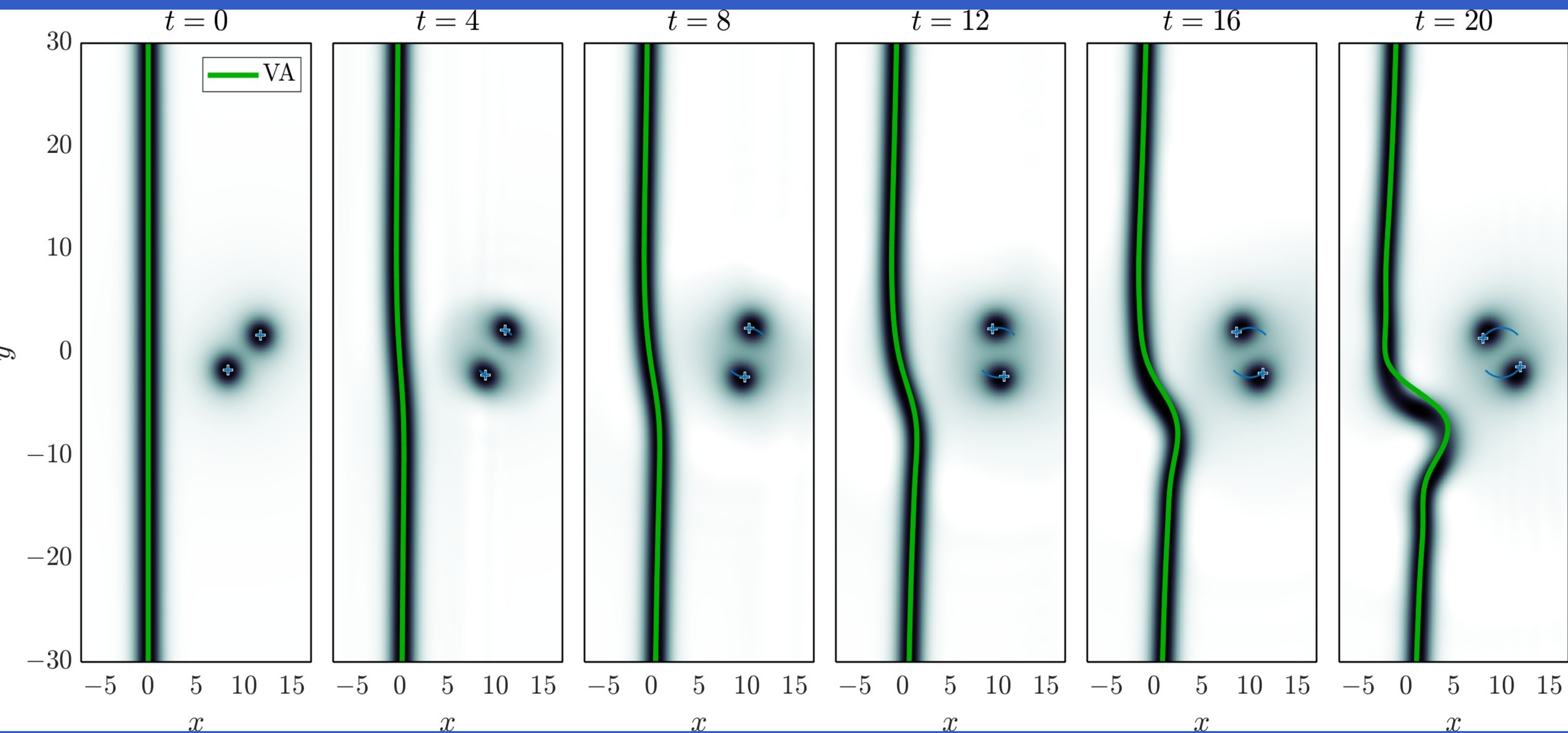
$$\begin{aligned} \dot{x}_m &= -\sum_{n \neq m}^{N} S_m \frac{y_m - y_n}{R_{mn}^2} \\ \dot{y}_m &= +\sum_{n \neq m}^{N} S_m \frac{x_m - x_n}{R_{mn}^2} \end{aligned}$$

# vortex $\to$ DSS + vortex $\leftrightarrow$ vortex: vortex dipole



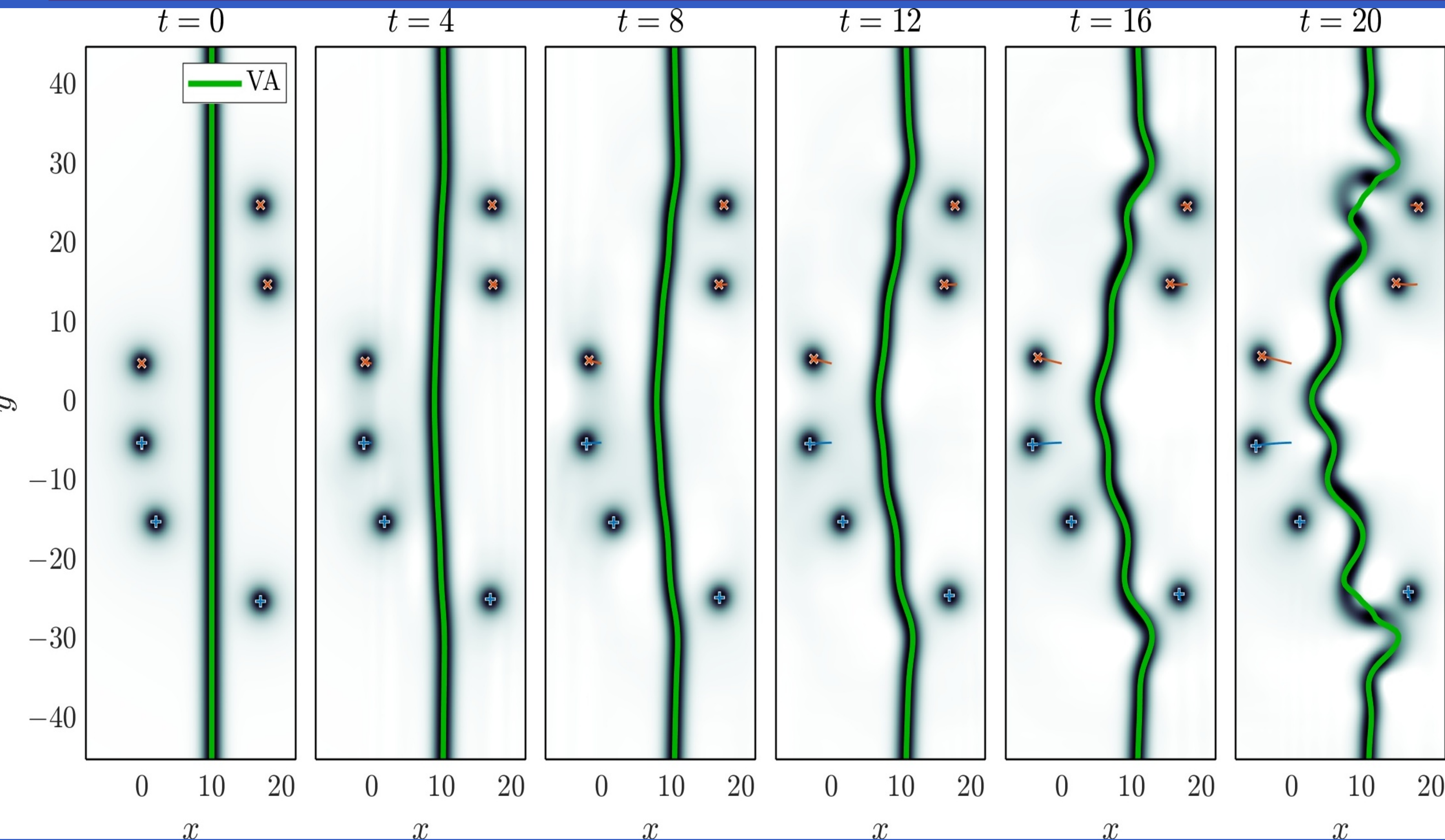

[movie]

# vortex → DSS + vortex ↔ vortex: corotating vortices




[movie]

# vortex → DSS + vortex ↔ vortex: vortex soup




[movie]

# Recap (inc. “not shown here”) and Outlook

- Solitonic filament dynamics (inc. *nonzero* external potentials):
  - Dark soliton stripes (NLS; AI)
  - Ring dark solitons (NLS; AI)
  - Dark-Bright solitons (NLS; AI)
  - Spherical shells in 3D (NLS; AI)
  - Kink (sG+$\phi^4$; AI)
  - Breather stripes (sG; VA)
  - Bright soliton stripes (inc. hyperbolic dispersion) (NLS; VA)

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  - Symmetric dark soliton-soliton interactions (NLS; AI)
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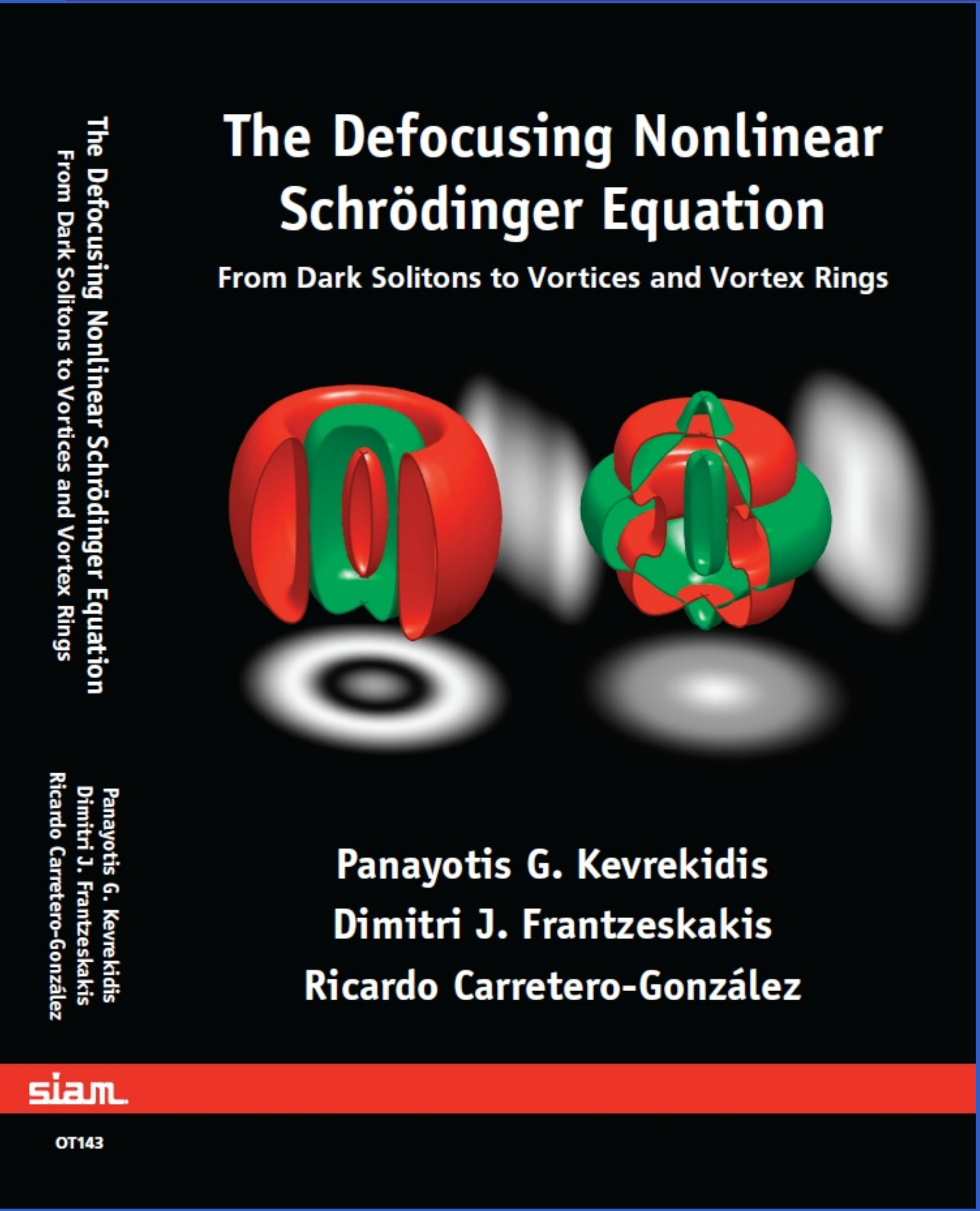
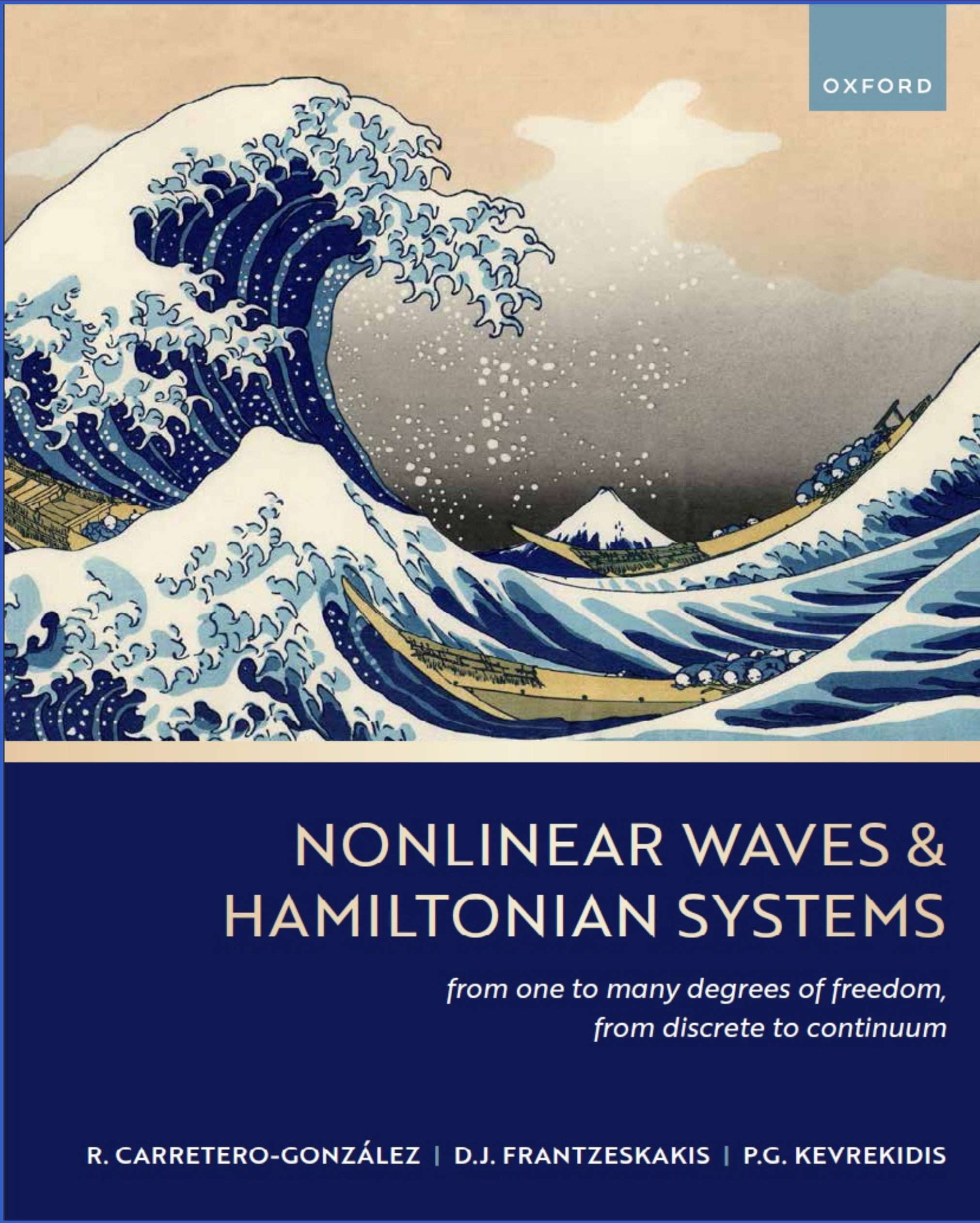
  - RDS-vortex interactions (NLS; AI)
  - Filament-vortex interactions (NLS; Pert+VA)
- Current Work and Outlook
  - Asymmetric filament-filament interactions ($x_1 \neq -x_2$)
  - Diff. Geometry $\rightarrow$ filaments on arclength/curvature/normal vel.
  - Transition between dark solitons and vortices
  - Include gain/loss $\rightarrow$ polariton BECs

# Details of these filament reductions:

- Kink-antikink interactions in the one- and two-dimensional sine-Gordon equation. Comm. Nonlinear Sci. Numer. Simulat. **109** (2022) 106123.
- Pairwise Interactions of Ring Dark Solitons with Vortices and other Rings: Stationary States, Stability Features and Nonlinear Dynamics. Phys. Rev. A **104** (2021) 023314
- Breather stripes and radial breathers of the two-dimensional sine-Gordon equation. Comm. Nonlinear Sci. Numer. Simulat. **94** (2021) 105596.
- Reduced dynamics for one and two dark soliton stripes in the defocusing nonlinear Schrödinger equation: a variational approach. Phys. Rev. Research **1** (2019) 033043.
- Dynamics of interacting dark soliton stripes. Phys. Rev. A **100** (2019) 033607.
- Dynamics and stabilization of bright soliton stripes in the hyperbolic-dispersion nonlinear Schrödinger equation. Comm. Nonlinear Sci. Numer. Simulat. **74** (2019) 268-281.
- Planar and Radial Kinks in Nonlinear Klein-Gordon Models: Existence, Stability and Dynamics. Phys. Rev. E **98** (2018) 052217.
- Adiabatic invariant analysis of dark and dark-bright soliton stripes in two-dimensional Bose-Einstein condensates. Phys. Rev. A **97** (2018) 063604.
- Adiabatic Invariant Approach to Transverse Instability: Landau Dynamics of Soliton Filaments. Phys. Rev. Lett. **118** (2017) 244101.

# SIAM, 2015: OUP, 2025:



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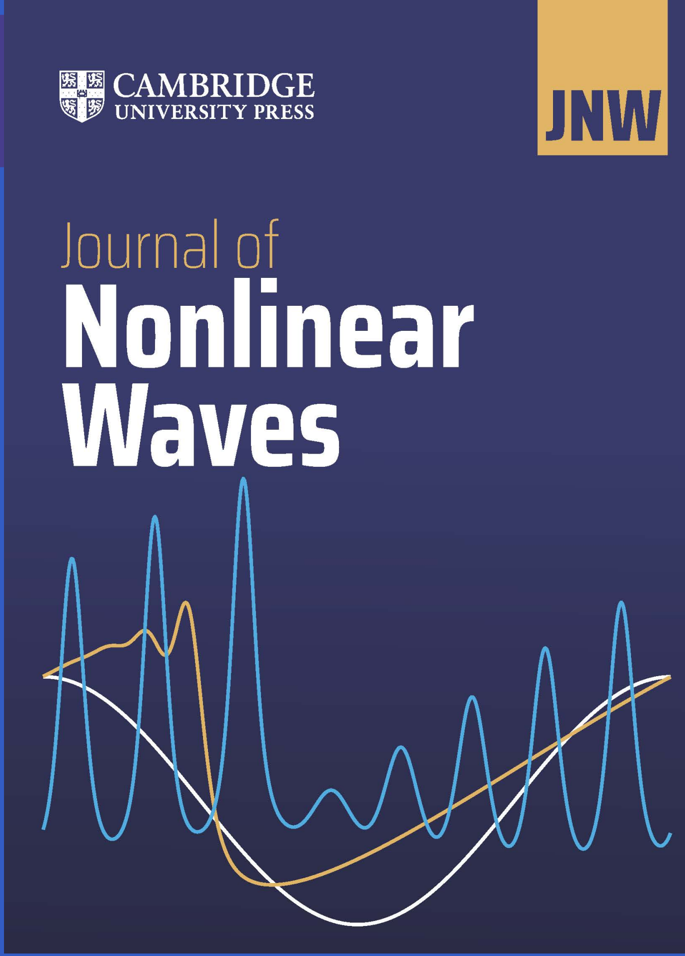


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  - MATH-636 : Mathematical Modeling
  - MATH-693A : Numerical Optimization
- Spring Year 1:
  - MATH-531 : Partial Differential Equations
  - MATH-638 : Dynamical Systems & Chaos II
  - MATH-693B : Numerical PDEs
- Fall Year 2:
  - MATH-635 : Pattern Formation
  - MATH-639 : Nonlinear Waves
  - MATH-797 : Research
- Spring Year 2:
  - MATH-799A : Thesis – Project