

Kinks, bright and dark solitons in the NLS equation with competing nonlinearities

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Joint works with P. G. Kevrekidis (Amherst), J. Holmer (Brown),
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The main object is the Hamiltonian NLS model with competing focusing and defocusing nonlinearities:

$$i\psi_t = \psi_{xx} + 4|\psi|^2\psi - 3|\psi|^4\psi.$$

The message is:

- ▶ there exists a kink solution separating bright and dark solitons;
- ▶ the kink is a degenerate minimizer of the conserved energy E ,
- ▶ nevertheless, after the right modulation, weighted metric, and energy splitting,

$$E(\psi(t)) - E(\phi) \gtrsim \rho_R(e^{i\alpha(t)}\psi(t, \cdot + \beta(t)), \phi),$$

which gives orbital stability of the kink with the profile ϕ .

- ▶ the kink persists with respect to external potentials.

The cubic–quintic model appears in two overlapping stories.

Optics. The quintic term is a higher-order correction to the nonlinear refractive index; bright and dark solitons have been studied both experimentally and analytically, for example in work of Akhmediev–Afanasjev (1995), Falcão-Filho et al. (2013), Reyna et al. (2020), Killip–Oh–Pocovnicu–Vişan (2017).

Quantum droplets. Lee–Huang–Yang corrections can compete with mean-field nonlinearities; Petrov (2015) and Petrov–Astrakharchik (2016) derived these corrections, and there is an active literature on droplets including Luo–Pang–Liu–Li–Malomed (2021), Katsimiga–Kevrekidis et al. (2023), Alfimov–Abdulaev (2025).

More general models with competing focusing and defocusing nonlinearities are used for similar applications.

The kink appears at the unit frequency, so that we can use the normalized equation

$$i\psi_t = \psi_{xx} - \psi + 4|\psi|^2\psi - 3|\psi|^4\psi.$$

It is Hamiltonian with conserved energy

$$E(\psi) = \int_{\mathbb{R}} (|\psi_x|^2 + |\psi|^2(1 - |\psi|^2)^2) \, dx.$$

The natural energy space is

$$\mathcal{E} = \{\psi \in H_{\text{loc}}^1(\mathbb{R}) : \psi_x \in L^2, \psi(1 - |\psi|^2) \in L^2\}.$$

Critical points of E satisfy

$$\phi'' - \phi + 4|\phi|^2\phi - 3|\phi|^4\phi = 0.$$

For real profiles, the ODE is a classical mechanical system

$$\phi'' + U'(\phi) = 0, \quad U'(\phi) = -\phi + 4\phi^3 - 3\phi^5,$$

where

$$U(\phi) = -\frac{1}{2}\phi^2 + \phi^4 - \frac{1}{2}\phi^6 = -\frac{1}{2}\phi^2(1 - \phi^2)^2.$$

The mechanical Hamiltonian is

$$H_{\text{mech}} = \frac{1}{2}(\phi')^2 + U(\phi).$$

The kink is the zero-energy heteroclinic orbit from $\phi = 0$ to $\phi = 1$.

Equivalently, $H_{\text{mech}} = 0$ yields

$$(\phi')^2 = \phi^2(1 - \phi^2)^2, \quad \phi' = \phi(1 - \phi^2) > 0.$$

The kink exists because of a delicate coefficient balance.

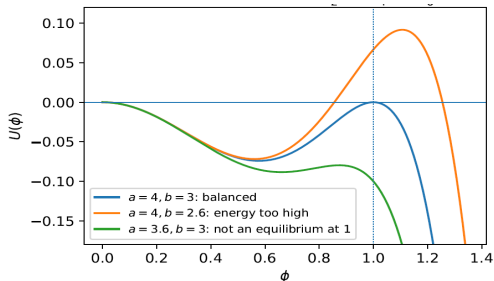
For the more general ODE

$$\phi'' - \phi + a\phi^3 - b\phi^5 = 0,$$

the mechanical Hamiltonian is

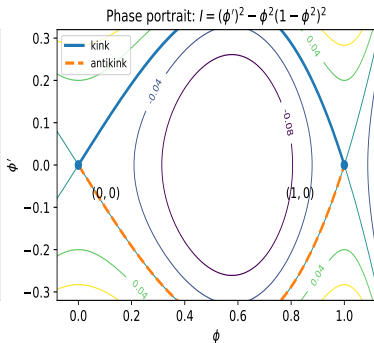
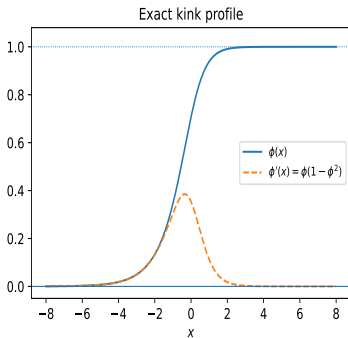
$$H_{\text{mech}} = \frac{1}{2}(\phi')^2 + U(\phi), \quad U(\phi) = -\frac{1}{2}\phi^2 + \frac{a}{4}\phi^4 - \frac{b}{6}\phi^6.$$

The soliton solutions include bright solitons, black solitons, and bubbles if $(a, b) \neq (4, 3)$. Only kinks exist for $(a, b) = (4, 3)$.



The explicit increasing kink is

$$\phi(x) = \left(\frac{1}{2}(1 + \tanh x) \right)^{1/2}, \quad \phi(-\infty) = 0, \quad \phi(+\infty) = 1.$$



Linearize about the real kink with $\psi = \phi + u + iv$. Then

$$\begin{cases} u_t = -\mathcal{L}_- v, \\ v_t = \mathcal{L}_+ u, \end{cases}$$

where

$$\mathcal{L}_- = -\partial_x^2 + 1 - 4\phi^2 + 3\phi^4, \quad \mathcal{L}_+ = -\partial_x^2 + 1 - 12\phi^2 + 15\phi^4.$$

Sturm theory gives nonnegativity:

$$\mathcal{L}_- \phi = 0, \quad \mathcal{L}_+ \phi' = 0, \quad \phi > 0, \quad \phi' > 0.$$

By the standard theory, **the kink is spectrally stable**. But $\mathcal{L}_-$ is not coercive in $H^1(\mathbb{R})$ since 0 is the end point of the continuous spectrum in $L^2(\mathbb{R})$. At the same time, $\mathcal{L}_+$ is coercive in $H^1(\mathbb{R})$ since 0 is an isolated eigenvalue from the continuous spectrum in $L^2(\mathbb{R})$. *The main issue is to turn the spectral stability of kinks into their nonlinear orbital stability.*

Theorem (Holmer–Kevrekidis–Pelinovsky, Proc AMS (2026))

Fix $R > R_0$. For every sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\psi_0 \in \mathcal{E} \subset L^\infty(\mathbb{R}), \quad \rho_R(\psi_0, \phi) < \delta^2,$$

then the corresponding global solution $\psi \in C^0(\mathbb{R}, \mathcal{E})$ satisfies

$$\inf_{\alpha, \beta \in \mathbb{R}} \rho_R(e^{i\alpha} \psi(t, \cdot + \beta), \phi) < \epsilon^2, \quad \forall t \in \mathbb{R}.$$

Here α is the phase modulation, β is the translation modulation, $\mathcal{E}$ is the energy space,

$$\mathcal{E} = \{\psi \in H_{\text{loc}}^1(\mathbb{R}) : \psi_x \in L^2, \psi(1 - |\psi|^2) \in L^2\}.$$

and $\rho_R(\cdot, \cdot)$ is the distance in $H^1(\mathbb{R})$ with the exponential weight on $[R, \infty)$, where $\phi(x) \rightarrow 1$ as $x \rightarrow +\infty$.

The Lyapunov functional is the conserved energy difference

$$\Lambda(\psi) := E(\psi) - E(\phi),$$

but coercivity of $\Lambda(\psi)$ at ϕ requires a modified split of the energy density.

Define the weighted space $\mathcal{H}_R$ by

$$\langle f, g \rangle_{\mathcal{H}_R} = \int_{-\infty}^R f \bar{g} \, dx + \int_R^{\infty} \frac{1 - \phi^2(x)}{1 - \phi^2(R)} f \bar{g} \, dx.$$

The orbital distance is

$$\rho_R(\psi, \phi) = \|\psi' - \phi'\|_{L^2}^2 + \|\psi - \phi\|_{\mathcal{H}_R}^2 + \| |\psi|^2 - |\phi|^2 \|_{L^2(R, \infty)}^2.$$

This metric sees the derivative globally, gives a weighted control of the complex perturbation on the nonzero background, and controls the density defect on (R, ∞) .

Following Gravejat–Smets (2015), and later variants by Alejo–Corcho (2024), Contreras–Pelinovsky–Plum (2018), and Pelinovsky–Plum (2024), we introduce

$$\eta = |\phi + u + iv|^2 - \phi^2 = 2\phi u + u^2 + v^2.$$

A direct expansion gives

$$E(\phi + u + iv) - E(\phi) = \int_{\mathbb{R}} [(u')^2 + (v')^2 + (1 - \phi^2)(1 - 3\phi^2)(u^2 + v^2) + (3\phi^2 - 2)\eta^2 + \eta^3] dx.$$

Equivalently,

$$E(\phi + u + iv) - E(\phi) = Q_-(u) + Q_-(v) + \int_{\mathbb{R}} [(3\phi^2 - 2)\eta^2 + \eta^3] dx,$$

where $Q_-(u)$ is a quadratic form related to $\mathcal{L}_-$ which is not coercive in $L^2(\mathbb{R})$ but is coercive in $\mathcal{H}_R$.

The problem: $3\phi^2 - 2 < 0$ on the left and $3\phi^2 - 2 > 0$ on the right.

Choose R with $\phi(R) > \sqrt{2/3}$. On $(-\infty, R)$ use the $\mathcal{L}_+$ quadratic density for u ; on (R, ∞) use the $\mathcal{L}_-$ density and the good η^2 term.

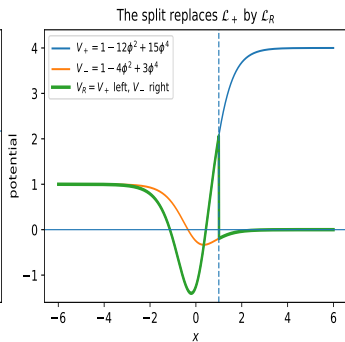
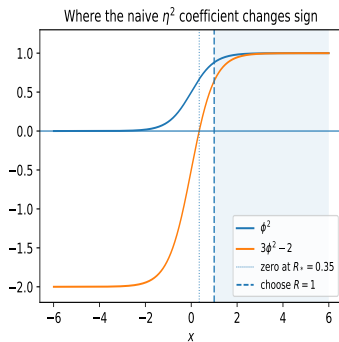
$$\begin{aligned} E(\phi + u + iv) - E(\phi) &= Q_R(u) + Q_-(v) + \int_R^{+\infty} [(3\phi^2 - 2)\eta^2 + \eta^3] dx \\ &\quad + \int_{-\infty}^R [(3\phi^2 - 2)(u^2 + v^2)(4\phi u + u^2 + v^2) + (2\phi u + u^2 + v^2)^3] dx \end{aligned}$$

This produces a piecewisely defined Schrödinger operator

$$\mathcal{L}_R = -\partial_x^2 + V_R,$$

where

$$V_R(x) = \begin{cases} 1 - 12\phi^2(x) + 15\phi^4(x), & x < R, \\ 1 - 4\phi^2(x) + 3\phi^4(x), & x > R. \end{cases}$$



The key coercivity estimates are:

$$Q_-(v) \geq C_- (\|v'\|_{L^2}^2 + \|v\|_{\mathcal{H}_R}^2), \quad \langle v, \phi \rangle_{\mathcal{H}_R} = 0$$

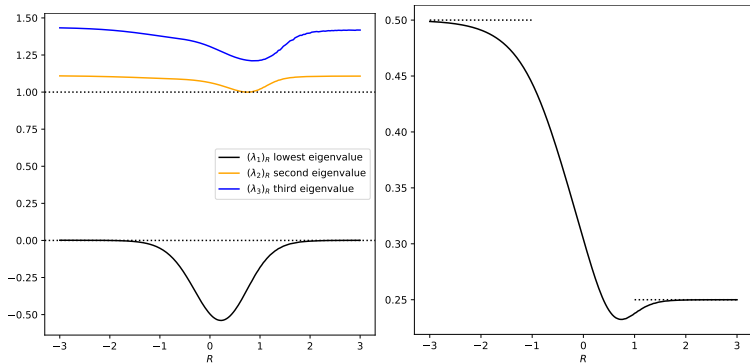
and

$$Q_R(u) \geq C_R (\|u'\|_{L^2}^2 + \|u\|_{\mathcal{H}_R}^2), \quad \langle u, \phi' \rangle_{L^2} = 0.$$

The coercivity estimate for $Q_-(v)$ follows from the fact that $\phi \in \mathcal{H}_R$, for which $\mathcal{L}_-\phi = 0$, and the continuous spectrum of $\mathcal{L}_-$ in $\mathcal{H}_R$ is $[1, \infty)$.

The coercivity estimate for $Q_R(u)$ is **more subtle since $\mathcal{L}_R$ has a small negative eigenvalue before imposing the constraint**. In particular, $\phi_R \in \mathcal{H}_R$, for which $Q_R(\phi_R) = 0$, is not in the domain space of $\mathcal{L}_R$:

$$\phi_R(x) = \begin{cases} \phi'(x), & x \in (-\infty, R), \\ \frac{\phi'(R)}{\phi(R)}\phi(x), & x \in (R, \infty). \end{cases}$$



Imposing the constraint

$$\langle u, \phi_R \rangle_{\mathcal{H}_R} = \langle u, \phi' \rangle_{L^2} = 0$$

yields coercivity of Q_R if and only if $\langle \mathcal{L}_R^{-1} \phi', \phi' \rangle_{L^2} < 0$. This can be proven for sufficiently large $R > 0$ or numerically for every R , see the right panel which shows $\lambda_1 \langle \mathcal{L}_R^{-1} \phi', \phi' \rangle_{L^2} > 0$ for every R .

The two orthogonality conditions are enforced by modulation.

If the solution remains close to the orbit, the implicit function theorem gives unique continuous parameters $\alpha(t)$ and $\beta(t)$ so that

$$e^{i\alpha(t)}\psi(t, \cdot + \beta(t)) = \phi + u(t) + iv(t),$$

with

$$\langle u(t), \phi' \rangle_{L^2} = 0, \quad \langle v(t), \phi \rangle_{\mathcal{H}_R} = 0.$$

The phase parameter removes the $\mathcal{L}_-$ neutral direction; the translation parameter removes the $\mathcal{L}_R$ neutral/negative direction.

With the modulation in place, conservation of E gives the Lyapunov estimate for small perturbations:

$$\begin{aligned} E(\psi_0) - E(\phi) &= E(\phi + u + iv) - E(\phi) \\ &\geq C \left(\|u' + iv'\|_{L^2}^2 + \|u + iv\|_{\mathcal{H}_R}^2 + \|\eta\|_{L^2(R, \infty)}^2 \right) \\ &\quad - \text{higher order terms.} \end{aligned}$$

The higher order terms are controlled by:

- ▶ Sobolev control on $(-\infty, R)$,
- ▶ the density variable η on (R, ∞) ,
- ▶ a bootstrap bound $\|\eta\|_{L^\infty(R, \infty)} \lesssim \epsilon_0$.

Thus $E(\psi_0) - E(\phi) \gtrsim \rho_R(\phi + u + iv, \phi)$ and the theorem is proven.

The same argument extends beyond cubic–quintic powers.

For $2 \leq p < q$, the normalized competing-power model is

$$i\psi_t = \psi_{xx} + \frac{q(p+2)}{2(q-p)}|\psi|^p\psi - \frac{p(q+2)}{2(q-p)}|\psi|^q\psi.$$

The kink appears again at the unit frequency and at the zero level curve

$$(\phi')^2 = \phi^2 \frac{q(1 - \phi^p) - p(1 - \phi^q)}{q - p}.$$

The same structure survives:

- ▶ weighted phase coercivity,
- ▶ constrained $\mathcal{L}_R$ coercivity,
- ▶ modulation by phase and translation,
- ▶ energy control in the ρ_R metric.

Discrete competing models also appear common in applications:

$$i \frac{du_n}{dt} + \varepsilon \Delta_2 u_n + |u_n|^p u_n - \gamma |u_n|^q u_n = 0, \quad n \in \mathbb{Z},$$

where $\gamma > 0$ is bifurcating parameter for standing waves of unit frequency $u(t) = e^{it} \hat{u}$.

- ▶ $\varepsilon \rightarrow 0$ is **the anticontinuum limit**, where many results can be derived with the explicit function theorem.

Alfimov–Korchagin–P, SIAM J. Appl. Dynam. Syst. (2026)

- ▶ $\varepsilon \rightarrow \infty$ is **the continuous limit**, where many results can be derived with the beyond-all-order asymptotics.

Lustri–Kevrekidis–Chapman, Quart. Appl. Math. (2025)

Lustri–Aniceto–Kevrekidis, SIAM J. Appl. Math. (2026)

Adriano–Hasmi–Kusdiantara–Susanto, Physica D (2025)

Theorem (Alfimov–Korchagin–Pelinovsky, SIAD (2026))

For every $1 \leq p < q$, there exists $\gamma_{p,q} > 0$ such that for every $\gamma \in (0, \gamma_{p,q})$ and small $\varepsilon \in (0, \varepsilon_0)$, there exists a bright soliton $u \in \ell^2(\mathbb{Z})$ satisfying

$$\|u - u^{(0)}\|_{\ell^2(\mathbb{Z})} \leq C_0 \varepsilon,$$

where $u^{(0)} = (\dots, 0, 0, \tilde{u}, 0, 0, \dots)$ with $\tilde{u} \in \mathbb{R}^N$ of any length $N \geq 1$ defined by nonzero roots $\pm a$ and $\pm A$ with $0 < a < A$ of the function $f(u) = u(1 - |u|^p + \gamma|u|^q)$ Moreover,

- (A) The solution u with either $\tilde{u} = (+A, +A, \dots, +A)$ or $\tilde{u} = (-A, -A, \dots, -A)$ is a local minimizer of energy.*
- (B) The solution u with either $\tilde{u} = (+a, -a, \dots, \pm a)$ or $\tilde{u} = (-a, +a, \dots, \mp a)$ is spectrally stable but it is a local constrained minimizer of energy for fixed mass Q iff $N = 1$.*

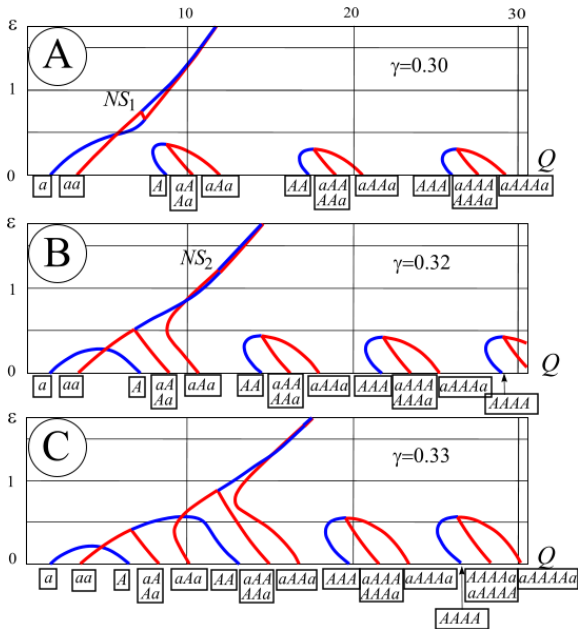


Figure: Bifurcations of branches of ILMs for the case $(p, q) = (2, 3)$.

The value $\gamma = \gamma_{p,q}$ is the bifurcation value where $a = A$ and the kink appears instead of the bright soliton in the standing waves of unit frequency $u(t) = e^{it}\hat{u}$.

For the cubic–quintic case,

$$\varphi(u_{n+1} - 2u_n + u_{n-1}) = u_n(1 - u_n^2)(1 - 3u_n^2), \quad n \in \mathbb{Z},$$

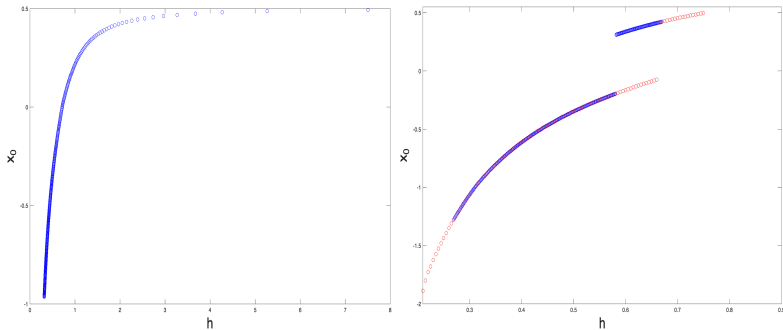
two distinct kinks appear in the anti-continuum limit:

$$\varepsilon = 0 : \quad u_n^{(1)} = \begin{cases} 0, & n \leq 0, \\ 1, & n \geq 1, \end{cases} \quad u_n^{(2)} = \begin{cases} 0, & n \leq -1, \\ \frac{1}{\sqrt{3}}, & n = 0, \\ 1, & n \geq 1, \end{cases}$$

$u^{(1)}$ is a stable minimizer of energy, $u^{(2)}$ is an unstable saddle point.
Adriano–Susanto, Stud. Appl. Math. (2026)

Do they persist in the continuum limit $\varepsilon \rightarrow \infty$?

Preliminary numerical results do not show persistence of any of the two kinks as $\varepsilon = h^{-1}$ grows (lattice spacing h is decreased).



Persistence of kinks is expected at the $\frac{1}{4}$ and $\frac{3}{4}$ of the lattice sites in the beyond-all-orders asymptotics.

Lustri–Kevrekidis–Pelinovsky, in progress (2026)

To be continued...