Some theoretical studies on the stochastic Gross-Pitaevskii equation

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Outline

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Dynamics of BEC at positive temperature

- At the zero temperature $T = 0$, all the atoms are well-presented by a single condensate function and the coherent evolution of the wave function is described by the standard Gross-Pitaevskii equation
- At higher temperature, all spontaneous and incoherent process (for ex. interaction with thermal cloud) may not be neglected
- The effect of such incoherent elements are implemented by adding a dissipation and a noise to the GP equation, called '(Simple Growth Projected) Stochastic GP equation' (P.B. Blakie et al., Advanced in Physics (2008))

$$d\psi = \mathcal{P}\Big\{ -\frac{i}{\hbar} L_{GP}\psi dt + \frac{\gamma}{k_B T}(\nu - L_{GP})\psi dt + dW(x,t)\Big\}$$

where $\mathcal{P}$: projection to the lowest energy modes, ν: chemical potential

$$L_{GP} = -\frac{\hbar^2}{2m}\nabla^2 + V(x) + g|\psi|^2, \langle dW(s,y), dW(t,x)\rangle = 2\gamma\delta_{t-s}\delta_{x-y}$$

This model is in very good qualitative agreement with the experiments aiming to simulate condensation process

- Spontaneous nucleation of vortices in BEC (Weiler et al. Nature (2008)).
  - Vortex formation occurrence during a phase transition predicted by Kibble-Zurek mechanism.
    Cf. Numerical simulations by Romain Poncet (2017).
- Investigation of the thermal (Gibbs) equilibrium, which gives the classification (Universality class) of the type of phase transitions (M.Kobayashi-L.Cugliandolo Phys. Rev. E (2016))

Remark that essentially, finite dimension models are analyzed in Physics.

Our aim: Study these models from a mathematical point of view, (i.e., in infinite dimension). In particular, interested in the Gibbs equilibrium.

We consider mathematically (for the moment in 1d):

- $(\Omega, \mathcal{F}, \mathbb{P})$: propability space endowed with filtration $(\mathcal{F}_t)_{t\geq 0}$
- The equation :

$$\begin{cases} dX & = (i+\gamma)(\partial_x^2 X - V(x)X + \nu X - \lambda |X|^2 X)dt + \sqrt{2\gamma} dW, \\ X(0) & = X_0, \quad t>0, \quad x \in \mathbb{R} \end{cases}$$

  where $\gamma > 0$. Assume $V(x) = x^2$, $\nu \geq 0$ and $\lambda = 1$ (defocusing).
- $(\partial_x^2 - x^2)h_k = -\lambda_k^2 h_k$ with $\lambda_k = \sqrt{2k+1}$, $k \in \mathbb{N}$. The eigenfunctions $h_k(x)$ are known as the Hermite functions.
- $W(t)$: cylindrical Wiener pocess on $L^2(\mathbb{R}, \mathbb{C})$, i.e.

$$W(t,x) = \sum_{k\in\mathbb{N}} \beta_k(t) h_k(x), \quad t>0, \ x \in \mathbb{R}$$

  where $\{\beta_k(t)\}_{k\in\mathbb{N}}$: a sequence of $\mathbb{C}$-valued independent Brownian motions.

Known results

- Burq, Tzvetkov and Thomann, ($\gamma = 0$, $\lambda = \pm 1$, $V(x) = x^2$, $1d$) Ann. Inst. Fourier (Grenoble) (2013)
  - Construction of Gibbs measure, Global Cauchy theory in the negative Sobolev space.
- Barton-Smith, ($\gamma \neq 0$, $\lambda = \pm 1$, $V(x) = 0$, bounded domain D, any d) Nonlinear differ. equ. appl.(2004)
  - Existence and Uniqueness of invariant measure on $L^p(D)$ ($p \in [2, \infty)$) for a not too small $\gamma \neq 0$
- E.A. Carlen, J. Fröhlich and J. Lebowitz, (regular noise, $\lambda = -1$, $V(x) = 0$, periodic boundary condi., $1d$) Commun. Math. Phys. (2016)
  - Construction of the "grand-canonical" Gibbs measure (i.e. modified Hamiltonian by a restoring term) and exponential convergence under some assumptions on the noise

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Gibbs measure

- Hamiltonian for the case of $\gamma = 0$: ($\nu = 0$ for simplicity)

$$H(u) = \frac{1}{2}\int_{\mathbb{R}} |(-\partial_x^2 + x^2)^{1/2} u|^2 dx + \frac{1}{4}\int_{\mathbb{R}} |u|^4 dx$$

- The Gibbs measure formally:

$$\begin{aligned} \rho(du) &= \Gamma e^{-H(u)} du \\ &= \Gamma e^{-\frac{1}{4}\int_{\mathbb{R}} |u|^4 dx} e^{-\frac{1}{2}((-\partial_x^2 + x^2)u, u)_{L^2}} du \end{aligned}$$

  where Γ is the normalizing constant.
- The last part may be written using the decomposition $u = \sum_k (a_k + ib_k) h_k$ with $(a_k, b_k) \in \mathbb{R}^2$,

$$\prod_k \frac{\lambda_k^2}{2\pi} e^{-\frac{\lambda_k^2}{2}(a_k^2 + b_k^2)} da_k db_k (=: \mu(du))$$

  This is a Gaussian measure, and can be interpreted as the law of random variable $\sum_{k\in\mathbb{N}} \frac{\sqrt{2}}{\lambda_k} g_k(\omega) h_k(x)$ with $\mathcal{L}(g_k) = \mathcal{N}_{\mathbb{C}}(0,1)$

Gibbs measure and the stationary solution

- Let us denote by

$$Z_\infty(t) = \sqrt{2\gamma}\int_{-\infty}^{t} e^{-(t-s)(i+\gamma)(-\partial_x^2+x^2)}dW(s),$$

the solution of

$$dZ = (i+\gamma)(\partial_x^2 - x^2)Zdt + \sqrt{2\gamma}dW$$

which is stationary.
- Write $Z_\infty(t)$ using the basis $\{h_k\}_k$,

$$Z_\infty(t) = \sqrt{2\gamma}\sum_{k\in\mathbb{N}}\Big(\int_{-\infty}^{t} e^{-(t-s)(i+\gamma)\lambda_k^2}d\beta_k(s)\Big)h_k$$

- The law of $Z_\infty(t)$ equals to the Gaussian measure μ, since

$$\mathcal{L}\Big(\sqrt{2\gamma}\int_{-\infty}^{0} e^{s(i+\gamma)\lambda_k^2}d\beta_k(s)\Big) = \mathcal{N}_{\mathbb{C}}\Big(0, \frac{2}{\lambda_k^2}\Big)$$

Support of the measure

- For $m \in \mathbb{N}$, $2m \geq p$, we have, by Minkowski's inequality,

$$|Z_\infty(t)|_{L^{2m}_\omega(L^p_x)} \leq C_m \Big| \sum_{k \in \mathbb{N}} \frac{|h_k(x)|^2}{\lambda_k^2} \Big|^{1/2}_{L^{p/2}_x} \leq C_m \Big(\sum_k \frac{|h_k(x)|^2_{L^p_x}}{\lambda_k^2} \Big)^{1/2}$$

- It is known the decay of h_k in L^p (Koch-Tataru, Duke Math.J. 2005): for $p \geq 4$, $|h_k|_{L^p(\mathbb{R})} \leq C_p \lambda_k^{-1/6}$, and by interpolation, if $2 \leq p \leq 4$, $|h_k|_{L^p(\mathbb{R})} \leq C_p \lambda_k^{-\frac{1}{3}(1-\frac{2}{p})}$.
- Recall that $\lambda_k^2 = 2k+1$ and the series converges for $p > 2$, i.e., $Z_\infty \in L^{2m}(\Omega; L^p)$ for any $m \geq p/2 > 1$; i.e. $Z_\infty \in L^p$ a.s. i.e. $\rho(L^p) = 1$ for $p > 2$.

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Let $p \geq 3$, $X_0 \in L^p(\mathbb{R})$, $\gamma > 0$ and $\nu = 0$ (Theorems hold also for $\nu > 0$).

Theorem

There exists a set $\mathcal{O} \subset L^p(\mathbb{R})$ such that $\rho(\mathcal{O}) = 1$, and such that for $X_0 \in \mathcal{O}$ there exists a unique solution $X(\cdot) \in C([0,\infty), L^p(\mathbb{R}))$ a.s.

$P_t\phi(y) := \mathbb{E}(\phi(X(t,y)))$, $y \in \mathcal{O}$, $t \geq 0$.

Theorem

Let $\phi \in L^2((L^p, d\rho), \mathbb{R})$, and $\bar{\phi} = \int_{L^p} \phi(y)d\rho(y)$. Then $P_t\phi(\cdot)$ converges exponentially to $\bar{\phi}$ in $L^2((L^p, d\rho), \mathbb{R})$, as $t \to \infty$; more precisely,

$$\int_{L^p} |P_t\phi(y) - \bar{\phi}|^2 d\rho(y) \leq e^{-\gamma t} \int_{L^p} |\phi(y) - \bar{\phi}|^2 d\rho(y).$$

Using Strong Feller property + Irreducibility of P_t on L^p,

Theorem

For any $X_0 \in L^p(\mathbb{R})$, there exists a unique solution $X(\cdot) \in C([0,\infty), L^p(\mathbb{R}))$ a.s.

Ideas for the proof

- Local existence in $L^p(\mathbb{R})$: Write $X = v + Z_\infty$ with v satisfying the deterministic PDE:

$$\partial_t v = (i+\gamma)(\partial_x^2 v - x^2 v - |v + Z_\infty|^2 (v + Z_\infty)), \ \ t > 0, \ \ x \in \mathbb{R}$$

  to obtain a sol. X in $C([0, T^*), L^p), p \geq 3$ (Ginibre-Velo, CMP. 1997) using the estimates on the linear semigroup obtained by Mehler's formula.

  Energy methods give a global bound in L^p, but it requires some restriction on the parameter γ. ("not too small $\gamma \neq 0$")
- Gibbs measure is invariant for the semigroup P_t.
- Globalization (in ρ- a.e. sense) by the invariance of Gibbs measure ρ supported on $L^p(\mathbb{R})$: Let $T > 0$, and $p \geq 3$. There exists C_T such that

$$\int_{L^p} \mathbb{E}\Big(\sup_{t \in [0, T^*)} |X(t, X_0)|_{L^p} \Big) \rho(dX_0) \leq C_T.$$

- Convergence to the equilibrium
  Key estimate: Poincaré inequality
  Let $\gamma > 0$. For any $\phi \in C_b^1(L^p)$, the following inequality is satisfied.

$$\int_{L^p} |\nabla_y \phi(y)|^2_{L^2_y} d\rho(y) \geq \int_{L^p} |\phi(y) - \bar{\phi}|^2 d\rho(y), \qquad (1)$$

  where $\bar{\phi} = \int_{L^p} \phi(y) d\rho(y)$, describes only measure's property, proved via a dissipative equation:

$$dY = \gamma(\partial_x^2 Y - x^2 Y - |Y|^2 Y)dt + \sqrt{2\gamma} dW.$$

  Gibbs measure $\rho(u) = e^{-H(u)} du$ is invariant for both dynamics

$$dX = (i + \gamma)(\partial_x^2 X - x^2 X - |X|^2 X)dt + \sqrt{2\gamma} dW.$$

  Important !
  - The balance $\gamma > 0$ between noise and dissipation
  - $H(u) = \frac{1}{2} \int_{\mathbb{R}} |(-\partial_x^2 + x^2)^{1/2} u|^2 dx + \frac{1}{4} \int_{\mathbb{R}} |u|^4 dx$ is same for both eqs.